THE BOUNDARY OF A DIVISIBLE CONVEX SET

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Abstract. We try to describe the boundary of a divisible convex set at an infinitesimal level. The geodesic flow of the Hilbert metric is the main tool in this study, because its asymptotic exponential behaviour (Lyapunov exponents) is related to the shape of the boundary of the convex set.

1. Introduction

I have studied in [Craar] the local asymptotic behaviour of the geodesic flow of Hilbert metrics. It naturally led me to introduce what seems to be a new regularity property of convex functions. I gave it the not-so-good name of *approximate regularity.*

Definition 1.1. *Let U be an open convex subset of $\mathbb{R}^{n-1}$ and $f : U \longrightarrow \mathbb{R}$ a $\mathcal{C}^1$ strictly convex function. For $x_0 \in U$ and small $v \in \mathbb{R}^{n-1}$, define $f_{x_0}(v) = f(x_0+v) - f(x_0) - d_{x_0}f(v)$. We say that the function f is* approximately regular *at the point $x_0 \in U$ if, for all $v \in \mathbb{R}^{n-1}$, the limit*

$$\lim_{t\to 0} \frac{\log(f_{x_0}(tv) + f_{x_0}(-tv))}{\log t}$$

exists.

The property is here defined for strictly convex $\mathcal{C}^1$ functions but it has a trivial extension to general convex functions. The main result of [Craar] about this property is the following decomposition theorem, that I proved using the geodesic flow of Hilbert metrics:

Theorem 1.2 ([Craar], Theorem 6.1)**.** *Let $f : U \longrightarrow \mathbb{R}$ be a $\mathcal{C}^1$ strictly convex function. The following propositions are equivalent:*

(i) *f is approximately regular at the point $x_0 \in U$;*

(ii) *there exist $1 \leqslant p \leqslant n-1$, a splitting $\mathbb{R}^{n-1} = \oplus_{i=1}^p G_i$ and numbers $+\infty \geqslant \alpha_1 > \cdots > \alpha_p \geqslant 1$ such that for all $v \in G_i$,*

$$\lim_{t\to 0} \frac{\log(f_{x_0}(tv_i) + f_{x_0}(-tv_i))}{\log t} = \alpha_i;$$

(iii) *there exist $1 \leqslant p \leqslant n-1$,a filtration $\{0\} = H_0 \subsetneq H_1 \subsetneq \cdots \subsetneq H_p = \mathbb{R}^{n-1}$ and numbers $+\infty \geqslant \alpha_1 > \cdots > \alpha_p \geqslant 1$ such that, for any $v_i \in H_i \smallsetminus H_{i-1}$,*

$$\lim_{t\to 0} \frac{\log(f_{x_0}(tv_i) + f_{x_0}(-tv_i))}{\log t} = \alpha_i.$$

When f is approximately regular at x_0, we call the numbers α_i the *Lyapunov exponents* of f at x_0. It will be more convenient in this work to count the Lyapunov exponents with multiplicities, taking into account the dimension of the subsets G_i. We thus define the vector of Lyapunov exponents $\alpha = (\alpha_i)_{i=1\cdots n-1}$, with $\alpha_1 \geqslant \cdots \geqslant \alpha_{n-1}$ and we say that f is *approximately α-regular* at x_0.

Apart from the previous theorem, I do not know what more can be said about approximate regularity. For example, I asked the question to know whether a convex function is Lebesgue-almost everywhere approximately regular, and to describe the range of possible Lyapunov exponents. But in fact, I do not even know if any convex function is approximately regular at at least one

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point.

In this note, I study these questions for the boundary of a divisible convex set for which lots of properties can be deduced from its numerous symmetries. As an example, let us give the following result.

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2. Hilbert geometry and divisible convex sets

2.1. **Hilbert geometry.** A *Hilbert geometry* is a metric space (Ω, d_Ω) where

- Ω is a *properly convex open set* of the real projective space $\mathbb{RP}^n$, $n \geqslant 2$; *properly* means that there exists a projective hyperplane which does not intersect the closure of Ω, or, equivalently, that there is an affine chart in which Ω appears as a relatively compact set;
- d_Ω is the distance on Ω defined, for two distinct points x, y, by

$$d_\Omega(x,y) = \frac{1}{2}|\log[a,b,x,y]|,$$

where a and b are the intersection points of the line (xy) with the boundary $\partial\Omega$ and $[a,b,x,y]$ denotes the cross ratio of the four points : if we identify the line (xy) with $\mathbb{R} \cup \{\infty\}$, it is defined by $[a,b,x,y] = \frac{|ax|/|bx|}{|ay|/|by|}$.

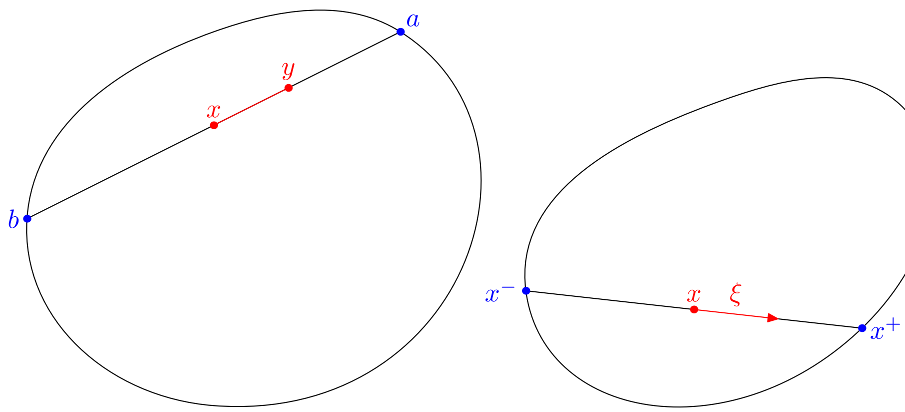

FIGURE 1. The Hilbert distance and the Finsler metric

These geometries had been introduced by Hilbert at the end of the nineteenth century as examples of spaces where lines are geodesics, which one can see as a motivation for the fourth of his famous problems: roughly speaking, this problem consisted in finding all geometries for which lines are geodesics.
When Ω is an ellipsoid, one recovers in this way the Beltrami model of the hyperbolic space. This is the only case where a Hilbert geometry is Riemannian. Otherwise, it is only a Finsler space: The Hilbert metric d_Ω is generated by a field of norms F on Ω, the norm $F(x,u)$ of a tangent vector $u \in T_x\Omega$ being given by the formula

$$F(x,u) = \frac{|u|}{2}\left(\frac{1}{|xu^+|} + \frac{1}{|xu^-|}\right),$$

where $|\ .\ |$ is an arbitrary Euclidean metric, and u^+ and u^- are the intersection points of the line $x + \mathbb{R}.u$ with the boundary $\partial\Omega$.

2.2. **Horospheres.** Assume Ω is strictly convex with $\mathcal{C}^1$ boundary. In this case, Busemann functions and horospheres can be defined as in hyperbolic geometry.
The *Busemann function* based at $x \in \partial\Omega$ is defined by

$$b_x(z,y) = \lim_{p \to x} d_\Omega(z,p) - d_\Omega(y,p),$$

which, in some sense, measures the (signed) distance from z to y in Ω as seen from the point $x \in \partial\Omega$.

The *horosphere* passing through $z \in \Omega$ and based at $x \in \partial\Omega$ is the set

$$\mathcal{H}_x(z) = \{y \in \Omega,\ b_x(z,y) = 0\}.$$

$\mathcal{H}_x(z)$ is also the limit when p tends to x of the metric spheres $S(p, d_\Omega(p,z))$ about p passing through z. In some sense, the points on $\mathcal{H}_x(z)$ are those which are as far from x as z is.

2.3. **Divisible convex sets.** Since projective transformations preserve cross-ratios, the group of projective symmetries of Ω,

$$\mathrm{Aut}(\Omega) = \{g \in \mathrm{PSL_{n+1}}(\mathbb{R}),\ g(\Omega) = \Omega\},$$

is a subgroup of isometries of the Hilbert geometry (Ω, d_Ω)[1]. A discrete subgroup Γ of $\mathrm{Aut}(\Omega)$ acts then properly discontinuously on Ω; by Selberg's lemma, it contains a finite index subgroup which has no torsion. The quotient $\Omega/_\Gamma$ is thus an orbifold in general, a manifold if Γ has no torsion.

Definition 2.1. *We say that a properly convex open set Ω or the corresponding Hilbert geometry (Ω, d_Ω) is* divisible *if there exists a discrete subgroup Γ of $\mathrm{Aut}(\Omega)$ with compact quotient $\Omega/_\Gamma$.*

The first example of divisible convex set is the ellipsoid, that is, the hyperbolic space. Y. Benoist proved the following alternative in [Ben04].

Theorem 2.2. *Let Ω be a divisible convex set, divided by a discrete subgroup Γ of $\mathrm{Aut}(\Omega)$. The following properties are equivalent:*

- *the convex set Ω is strictly convex;*
- *the boundary $\partial\Omega$ is of class $\mathcal{C}^1$;*
- *the Hilbert geometry (Ω, d_Ω) is Gromov-hyperbolic;*
- *the group Γ is Gromov-hyperbolic.*

An important argument of duality is used to prove this theorem, that we recall now. Consider one of the two convex cones $C \subset \mathbb{R}^{n+1}$ whose trace is Ω. The dual convex set Ω^* is the trace of the dual cone

$$C^* = \{f \in (\mathbb{R}^{n+1})^*,\ \forall x \in C,\ f(x) > 0\}.$$

The set Ω^* can be identified with the set of projective hyperplanes which do not intersect $\overline{\Omega}$: to such a hyperplane corresponds the line of linear maps whose kernel is the given hyperplane. For example, we can see the boundary of Ω^* as the set of tangent spaces to $\partial\Omega$. In particular, when Ω is strictly convex with $\mathcal{C}^1$ boundary, there is a homeomorphism between the boundaries of Ω and Ω^*: to the point $x \in \partial\Omega$ we associate the (projective class of the) linear map x^* such that $\ker x^* = T_x\partial\Omega$. The group $\mathrm{Aut}(\Omega)$ acts on the dual convex set Ω^* via $g.y = ({}^t g)^{-1}(y)$, $g \in \mathrm{Aut}(\Omega)$.

Lemma 2.3 ([Ben04], Lemme 2.8)**.** *Let Γ be a discrete subgroup of $\mathrm{Aut}(\Omega)$. The action of Γ on Ω is cocompact if and only if the action of Γ on Ω^* is also cocompact.*

[1] It is conjectured that, for most Hilbert geometries, all isometries are projective.

Apart from the ellipsoid, various examples of strictly convex divisible sets have been given. Some can be constructed using Coxeter groups ([KV67], [Ben06b]), some by deformations of hyperbolic manifolds (based on [JM87] and [Kos68], see also [Gol90] for the 2-dimensional case); we should also quote the exotic examples of M. Kapovich [Kap07] of divisible convex sets in all dimensions which are not quasi-isometric to the hyperbolic space (Y. Benoist [Ben06b] had already given an example in dimension 4).
Non-strictly convex examples are more difficult to find. The trivial ones are given by the symmetric spaces of the groups $\mathrm{SL}_n(\mathbb{K})$ ($\mathbb{K}$ being the set of complex, quaternionic or octonionic numbers[2]) or by products (see the historical remarks in [Ben03]). The only other known examples have been constructed by Y. Benoist [Ben06a] and L. Marquis [Mar10] in dimension 3 using Coxeter groups.

2.4. **Properties of the dividing group.** Let $\Omega \subset \mathbb{RP}^n$ be a properly convex strictly convex set, divided by a torsion-free discrete group Γ. All elements $g \in \Gamma$ are *hyperbolic isometries* of the Hilbert geometry (Ω, d_Ω). That means the following.
The element g fixes exactly two points x_g^+ and x_g^- on $\partial\Omega$; the point x_g^+ is the attractive point of g, x_g^- is the repulsive point of g : for any point $x \in \overline{\Omega} \smallsetminus \{x_g^-, x_g^+\}$, $\lim_{n\to\pm\infty} g^n(x) = x_g^\pm$.
Denote by $(\ell_i(g))_{i=0\cdots n}$ the complex eigenvalues of g, counted with multiplicities and ordered such that $|\ell_0(g)| \geqslant |\ell_1(g)| \geqslant \cdots \geqslant |\ell_n(g)|$. The largest and smallest eigenvalues ℓ_0 and $\ell_n(g)$ are simple, real and positive, and the points x_g^+ and x_g^- are the corresponding eigenvectors.
Let $\lambda_i(g) = \log|\ell_i(g)|$, $i = 0\cdots n$. The isometry g acts as a translation of length $\frac{1}{2}(\lambda_n(g) - \lambda_0(g))$ on the open segment $]x_g^- x_g^+[$.

The following result will be crucial to deduce some rigidity results.

Theorem 2.4 (Y. Benoist [Ben00])**.** *Let $\Omega \subset \mathbb{RP}^n$ be a properly convex strictly convex set, divided by a discrete group Γ. The group Γ is Zariski-dense in $\mathrm{SL_{n+1}}(\mathbb{R})$, unless Ω is an ellipsoid.*

Recall that the *Zariski-closure* of a subgroup Γ of $\mathrm{SL_{n+1}}(\mathbb{R})$ is the smallest algebraic subgroup G of $\mathrm{SL_{n+1}}(\mathbb{R})$ which contains Γ. We then say that Γ is *Zariski-dense* in G.
The hypothesis of strict convexity in the last theorem is actually unnecessary, but the proof in the general case is far more involved [Ben03].
This last theorem will be useful through the following characterization of Zariski-dense subgroups of semisimple Lie groups, which is also due to Y. Benoist, and that we explain in the case of the group $\mathrm{SL_{n+1}}(\mathbb{R})$. To each element g in $\mathrm{SL_{n+1}}(\mathbb{R})$, we associate the vector $\log(g) = [\lambda_0(g) : \cdots : \lambda_n(g)] \in \mathbb{RP}^n$ and for a subgroup Γ of $\mathrm{SL_{n+1}}(\mathbb{R})$, we set $\log\Gamma = \overline{\{\log g,\ g \in \Gamma\}}$.

Theorem 2.5 (Y. Benoist, [Ben97])**.** *Let Γ be a subgroup of $\mathrm{SL_{n+1}}(\mathbb{R})$. If Γ is Zariski-dense in $\mathrm{SL_{n+1}}(\mathbb{R})$, then $\log\Gamma$ has nonempty interior.*

3. Curvature of the boundary

3.1. **What is curvature.** Let us begin with an old theorem of A. D. Alexandrov [Ale39] about convex functions:

Theorem 3.1. *Let $f : U \subset \mathbb{R}^{n-1} \longmapsto \mathbb{R}$ be a convex function defined on a convex open set U of $\mathbb{R}^{n-1}$. The Hessian matrix $\mathrm{Hess}(f) = \left(\frac{\partial^2 f}{\partial_i \partial_j}\right)_{ij}$ exists Lebesgue almost everywhere in U.*

Let Ω be a bounded convex set of the Euclidean space $\mathbb{R}^n$. It is then possible to compute the Hessian of its boundary at Lebesgue almost every point $x \in \partial\Omega$. We will call a $\mathcal{C}^2$ *point* a point x where this is possible.
The Hessian is a positive symmetric bilinear form on the tangent space $T_x\partial\Omega$. It represents the curvature of the boundary at x. When it is degenerate, that means the curvature of the boundary is zero in some tangent direction.

[2] In the case of octionions, the only possibility is $n = 3$.

The Hessian is a Euclidean notion, but its degeneracy is not. Namely, if Ω is a properly convex open set of $\mathbb{RP}^n$ and x a point of $\partial\Omega$, we can choose an affine chart centered at x and a metric on it and compute the Hessian of $\partial\Omega$ at x; its degeneracy does not depend on the choice of the affine chart and the metric.

We can *measure* the vanishing of the curvature of $\partial\Omega$ in the following way. Fix a smooth measure λ^* on the boundary of the dual convex set Ω^*, and call λ its pull-back to $\partial\Omega$. Then λ can be seen as a measure of the curvature of $\partial\Omega$. It can be decomposed as

$$\lambda = \lambda^{ac} + \lambda^{sing},$$

where λ^{ac} is an absolutely continuous measure and λ^{sing} is singular with respect to any Lebesgue measure on $\partial\Omega$. For example, in dimension 2, if $\partial\Omega$ is not $\mathcal{C}^1$ at some point x then λ will have an atom at x. The support of λ^{ac} is the closure of the set of $\mathcal{C}^2$ points with nondegenerate Hessian. Though Ω is convex, it may happen that $\lambda^{ac} = 0$, that is, λ is singular with respect to some (hence any) smooth measure on $\partial\Omega$. This is equivalent to the fact that the Hessian is degenerate at Lebesgue-*almost all* $\mathcal{C}^2$ point of $\partial\Omega$. We then say that *the curvature of the boundary is supported on a set of zero Lebesgue measure.*

3.2. **Curvature of the boundary of a divisible convex set.** The curvature of the boundary of a divisible convex set has been investigated by J.-P. Benzécri [Ben60].

Lemma 3.2 (J.-P. Benzécri [Ben60])**.** *Let X_n denote the set of properly convex open sets of $\mathbb{RP}^n$, equipped with the Hausdorff topology. Let $\Omega \in X_n$.*

- *If there exists a $\mathcal{C}^2$ point $x \in \partial\Omega$ with nondegenerate Hessian, then the closure of the orbit $\mathrm{PSL_{n+1}}(\mathbb{R}) \cdot \Omega$ in X_n contains an ellipsoid.*
- *If Ω is divisible then the orbit $\mathrm{PSL_{n+1}}(\mathbb{R}) \cdot \Omega$ is closed in X_n.*

Proof. These two results are respectively Propositions 5.3.10 and 5.3.3 of [Ben60]. Let us recall the proofs.
Choose an affine chart and a Euclidean metric on it such that Ω appears as a bounded convex open set of $\mathbb{R}^n$. Let x be a point of $\partial\Omega$ with nondegenerate Hessian. Let $\mathcal{E}$ be the osculating ball of $\partial\Omega$ at x. It defines a hyperbolic geometry $(\mathcal{E}, d_{\mathcal{E}})$. Pick a point $y \in \partial\mathcal{E}$ distinct from x, and choose a hyperbolic isometry g of $\mathcal{E}$ whose attracting fixed point y and repulsive one x. Now, since $\partial\mathcal{E}$ and $\partial\Omega$ are tangent up to order 2, it is not difficult to see that $g^n \cdot \Omega$ converges to $\mathcal{E}$ when n goes to $+\infty$. This proves the first point.
The second point is a consequence of another result of Benzécri, which says that the action of $\mathrm{PSL_{n+1}}(\mathbb{R})$ on the set $\dot{X}_n = \{(\Omega, x),\ \Omega \in X_n, x \in \Omega\}$ is proper (this is Théorème 3.2.1 of [Ben60]). Each orbit $\mathrm{PSL_{n+1}}(\mathbb{R}) \cdot (\Omega, x)$ is thus closed. Now, the orbit $\mathrm{PSL_{n+1}}(\mathbb{R}) \cdot \Omega$ is closed in X_n if and only if the union $\cup_{x\in\Omega} \mathrm{PSL_{n+1}}(\mathbb{R}) \cdot (\Omega, x)$ is closed in $\dot{X}_n$. Since Ω is divisible, divided, say, by the group Γ, there is a compact subset K of Ω such that $\Gamma \cdot K = \Omega$. So the union

$$\bigcup_{x\in\Omega} \mathrm{PSL_{n+1}}(\mathbb{R}) \cdot (\Omega, x) = \bigcup_{x\in K} \bigcup_{g\in\Gamma} \mathrm{PSL_{n+1}}(\mathbb{R}) \cdot (\Omega, g(x)) = \bigcup_{x\in K} \mathrm{PSL_{n+1}}(\mathbb{R}) \cdot (\Omega, x)$$

is closed in $\dot{X}_n$. □

More about Benzécri's contributions can be found in L. Marquis's survey [Mar13]; the proof of the second point above is actually taken from it. As a consequence of the last lemma, we get the following

Proposition 3.3. *Let $\Omega \subset \mathbb{RP}^n$ be a divisible convex set, and assume Ω is not an ellipsoid. Then any $\mathcal{C}^2$ point has degenerate Hessian. In particular, the curvature of $\partial\Omega$ is supported on a subset of zero Lebesgue measure.*

Proof. Assume that the Hessian of $\partial\Omega$ is not degenerate at some $\mathcal{C}^2$ point. Lemma 3.2 implies that the orbit $\mathrm{PSL_{n+1}}(\mathbb{R}) \cdot \Omega$ is closed and contains an ellipsoid. So Ω itself is an ellipsoid. □

When the convex set is strictly convex, the geodesic flow of the Hilbert metric allows to say more about the properties of the boundary. The rest of this paper is dedicated to this case.

4. The geodesic flow and the boundary

4.1. **The geodesic flow.** When Ω is strictly convex, the metric space (Ω, d_Ω) is uniquely geodesic, and the geodesics are lines. The geodesic flow is then well defined on the homogeneous bundle $\pi : H\Omega \longrightarrow \Omega$ of tangent directions: To find the image by φ^t of a point $w = (x, [\xi]) \in H\Omega$, consisting of a point and a direction, one follows the geodesic line c_w leaving x in the direction $[\xi]$, and one has $\varphi^t(w) = (c_w(t), [c'_w(t)])$.
By projection, this also defines the geodesic flow on HM, the homogeneous bundle of $M = \Omega/_{\Gamma}$. The geodesic flow has the same regularity as the boundary of Ω. So, if Ω is strictly convex and divisible, by Theorem 2.2, it is a $\mathcal{C}^1$ flow. We will denote by X the generator of the geodesic flow (both on $H\Omega$ or HM).

Theorem 4.1 (Y. Benoist, [Ben04])**.** *Let $M = \Omega/_{\Gamma}$ a compact manifold quotient of a strictly convex set $\Omega \subset \mathbb{RP}^n$. The geodesic flow on HM is an Anosov flow: There exist a φ^t-invariant splitting of the tangent bundle*

$$THM = \mathbb{R}.X \oplus E^u \oplus E^s$$

and constants $C, \alpha > 0$ such that, for any $t \geqslant 0$,

$$\|d\varphi^t(Z^s)\| \leqslant Ce^{-\alpha t}\|Z^s\|,\ Z^s \in E^s,$$

$$\|d\varphi^{-t}(Z^u)\| \leqslant Ce^{-\alpha t}\|Z^u\|,\ Z^u \in E^u.$$

Here the norm $\|\cdot\|$ denotes an arbitrary Finsler metric on HM; because HM is compact, the Anosov property of the flow does not depend on the metric, even if the constants C and/or a do.
In our situation, the stable and unstable bundles E^s and E^u can be geometrically understood using horospheres. For $w \in H\Omega$, define the sets

$$W^s(w) = \{v \in H\Omega \mid v^+ = w^+,\ \pi(v) \in \mathcal{H}_{w^+}(\pi(w))\},$$

and

$$W^u(w) = \{v \in H\Omega \mid v^- = w^-,\ \pi(v) \in \mathcal{H}_{w^-}(\pi(w))\}.$$

The sets $W^s(w)$ and $W^u(w)$ are $\mathcal{C}^1$ submanifolds of $H\Omega$ and it is not difficult to see that they are the stable and unstable sets of the geodesic flow (d denotes the distance generated by $\|\cdot\|$):

$$W^s(w) = \{v \in H\Omega,\ \lim_{t\to+\infty} d(\varphi^t w, \varphi^t v) = 0\}$$

and

$$W^u(w) = \{v \in H\Omega,\ \lim_{t\to-\infty} d(\varphi^t w, \varphi^t v) = 0\}.$$

Both families $W^s(w), w \in H\Omega$ and $W^u(w), w \in H\Omega$ form a φ^t-invariant foliation of $H\Omega$. Everything projects down on HM where we will use the same notation. The stable and unstable bundles are then the tangent spaces to the stable and unstable foliations : $E^s(w) = T_wW^s(w)$, $E^u(w) = T_wW^u(w)$.

The asymptotic behaviour of the geodesic flow is encoded in the boundary of Ω: When we look at the behaviour of the norm $\|d\varphi^t Z\|$ when t goes to $+\infty$, for some $Z \in T_wH\Omega$, we see appearing naturally the graph of the boundary at the extremal point w^+. This observation is at the basis of this work. To illustrate this observation, notice the following "consequence" of Theorem 4.1:

Proposition 4.2 ([Ben04], Proposition 4.6)**.** *The boundary of a divisible strictly convex set is $\mathcal{C}^\alpha$ and β-convex for some $1 < \alpha \leqslant 2$, $\beta \geqslant 2$. In particular, the geodesic flow is $\mathcal{C}^\alpha$ for some $\alpha > 1$.*

To understand the last statement, we recall the following definitions:

Definition 4.3. *Let $1 < \alpha < 2, \beta > 1$ and U an open subset of $\mathbb{R}^n$. A $\mathcal{C}^1$-function $f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ is*

- *of class $\mathcal{C}^\alpha$ if, for some constant $C > 0$,*
$$|f(x) - f(y) - d_x f(y-x)| \leqslant C|x-y|^{1+\varepsilon},\ x, y \in U;$$
- *β-convex if, for some constant $C > 0$,*
$$|f(x) - f(y) - d_x f(y-x)| \geqslant C|x-y|^{\beta},\ x, y \in U.$$

4.2. **Approximate regularity and Lyapunov exponents.** Recall the following definition:

Definition 4.4. *Let $\Omega \subset \mathbb{RP}^n$ be a strictly convex set with $\mathcal{C}^1$ boundary. A point $w \in H\Omega$ is* weakly regular *if, for any $Z \in T_w H\Omega \smallsetminus \{0\}$, the limit*
$$\chi(Z) = \lim_{t \to \pm\infty} \frac{1}{t} \log \|d\varphi^t(Z)\|$$
exists. It is said to be forward *or* backward weakly regular *if the limits exist only when t goes to $+\infty$ or $-\infty$ (or if both limits differ). The number $\chi(Z)$ is called the Lyapunov exponent of Z.*

Because stable and unstable manifolds $W^s(w)$ and $W^u(w)$ at w have the same projection on Ω, there is a symmetry between the action of the flow on stable and unstable vectors (see for example Lemma 2.3 in [Craar]). In particular, we can see that if $Z^s \in E^s(w), Z^u \in E^u(w)$ project on the same vector $z \in \mathcal{H}_w$, then
$$\chi(Z^u) = 2 + \chi(Z^s).$$
The complete behaviour is then encoded in the behaviour of unstable vectors, and we will be only interested in these vectors by looking at the restriction of the differential $d\varphi^t$ to the bundle E^u. Because the geodesic flow is an Anosov flow, all Lyapunov exponents of unstable vectors are positive.

Given a forward weakly regular point $w \in H\Omega$, the numbers $\chi(Z)$, for any $Z \in E^u(w)$, can take only a finite number $0 < \chi_1 < \cdots < \chi_p$ of values, which are called the *positive Lyapunov exponents* of w. There is then a φ^t-invariant splitting
$$TH\Omega = E_1 \oplus \cdots \oplus E_p$$
along the orbit $\varphi.w$, called *Lyapunov splitting*, such that, for any vector $Z_i \in E_i \smallsetminus \{0\}$,
$$\lim_{t \to +\infty} \frac{1}{t} \log \|d\varphi^t(Z_i)\| = \chi_i.$$
As for the exponents (α_i) appearing in the definition of approximate regularity, we will count the (χ_i) with multiplicities. We thus have $n-1$ positive Lyapunov exponents $(\chi_i)_{i=1\cdots n-1}$ ordered as $\chi_1 \leqslant \cdots \leqslant \chi_n$. The main result of [Craar] is the following

Theorem 4.5 ([Craar], Theorem 1)**.** *Let $\Omega \subset \mathbb{RP}^n$ be a strictly convex set with $\mathcal{C}^1$ boundary. A point $w \in H\Omega$ is forward weakly regular if and only if the boundary $\partial\Omega$ is approximately regular at the point $w^+ = \varphi^{+\infty}(w)$. If $0 \geqslant \chi_1 \leqslant \cdots \leqslant \chi_n$ are the positive Lyapunov exponents of w, then $\partial\Omega$ is approximately α-regular with $\alpha = (\alpha_i)_{i=1\cdots n-1}$ given by $\alpha_i = 2/\chi_i$.*

5. The set of approximately regular points and the range of Lyapunov exponents

Let $\Omega \subset \mathbb{RP}^n$ be a divisible strictly convex set. Our interest now will lie on the set of approximately regular points $\Lambda \subset \partial\Omega$, as well as the set of all possible Lyapunov exponents
$$\mathcal{A} = \{\alpha(x) \in \mathbb{R}^{n-1},\ x \in \Lambda\}.$$
By projective invariance of the notion of approximate-regularity, the set of approximately α-regular points of $\partial\Omega$ is Γ-invariant, for any vector α. Since the action of Γ on $\partial\Omega$ is minimal, it is either empty (if $\alpha \notin \mathcal{A}$) or a dense subset of $\partial\Omega$ (if $\alpha \in \mathcal{A}$).

5.1. **Oseledets' theorem.** The following result is a version of Osedelets' ergodic multiplicative theorem [Ose68]:

Theorem 5.1. *Let $M = \Omega/_\Gamma$ a manifold quotient of a strictly convex set $\Omega \subset \mathbb{RP}^n$ with $\mathcal{C}^1$ boundary. Let μ a φ^t-invariant probability measure on HM. The set of weakly regular points has full μ-measure.*

It allows us to deduce the following

Corollary 5.2. *Let $\Omega \subset \mathbb{RP}^n$ be a divisible strictly convex set. The set $\mathcal{A}$ is nonempty and the set Λ is dense in $\partial\Omega$.*

Proof. The set of φ^t-invariant probability measures on HM is nonempty. In particular, by Oseledets's theorem, there exists a weakly regular point $w \in H\Omega$. By Theorem 4.5, the boundary $\partial\Omega$ is approximately regular at the point w^+, so $\mathcal{A}$ is nonempty and Λ is dense in $\partial\Omega$. □

5.2. **Hyperbolic isometries and closed orbits.** Recall that any element $g \in \Gamma$ is a hyperbolic isometry of the Hilbert geometry (Ω, d_Ω). We use the notation introduced in section 2.4.
For $g \in \Gamma$, pick a point $w \in H\Omega$ such that $w^- = x_g^-,\ w^+ = x_g^+$. The projection on HM of the orbit of w under the flow is a closed orbit of the flow, of length $\frac{1}{2}(\lambda_n(g) - \lambda_0(g))$. Two elements g and g' yield the same closed orbit if and only if they are conjugated. Conversely, any closed orbit is obtained in this way. In other words: Closed orbits of the geodesic flow on HM are in bijection with conjugacy classes of $\Gamma \smallsetminus \{1\}$.

Proposition 5.3. *Let $\Omega \subset \mathbb{RP}^n$ be a divisible strictly convex set, divided by a torsion-free discrete group $\Gamma < \mathrm{Aut}(\Gamma)$. Let $g \in \Gamma$. The boundary $\partial\Omega$ is approximately $\alpha(g)$-regular at the point x_g^+, with $\alpha(g) = (\alpha_i(g))_{i=1\cdots n-1}$ given by*

$$\alpha_i(g) = \frac{1 - \lambda_n(g)/\lambda_0(g)}{1 - \lambda_i(g)/\lambda_0(g)}. \tag{5.1}$$

Proof. In [Cra09], I showed that the positive Lyapunov exponents $(\chi_i(g))_{i=1\cdots n-1}$ of the closed orbit corresponding to the (conjugacy class of the) element $g \in \Gamma$ were given by

$$\chi_i(g) = 2\,\frac{\lambda_0(g) - \lambda_i(g)}{\lambda_0(g) - \lambda_n(g)}.$$

Theorem 4.5 gives the result. □

The element $g \in \Gamma$ acts on the dual convex set Ω^* by $g.y = ({}^tg)^{-1}(y)$. To $g \in \Gamma$, we thus associate the isometry $g^* = ({}^tg)^{-1} \in \mathrm{Aut}(\Omega^*)$. The dual point to x_g^+ is the point $x_{g^*}^-$, at which $\partial\Omega^*$ is approximately $\alpha(g^*)$-regular, with $\alpha(g^*) = (\alpha_i(g^*))_{i=1\cdots n-1}$ given by

$$\alpha_i(g^*) = \frac{1 - \lambda_n(g)/\lambda_0(g)}{1 - \lambda_n(g)/\lambda_{n-i}(g)}.$$

Remark that

$$\frac{1}{\alpha_{n-i}(g^*)} + \frac{1}{\alpha_i(g)} = 1,\ i = 1\cdots n-1.$$

In general, if $\partial\Omega$ is approximately α-regular at some point x with $\alpha = (\alpha_i)_{i=1\cdots n-1}$, one can expect $\partial\Omega$ to be approximately α^*-regular at the dual point $x^* \in \partial\Omega^*$ with $\alpha^* = (\alpha_i^*)_{i=1\cdots n-1}$ satisfying to the previous relation: $1/\alpha_{n-i}^* + 1/\alpha_i = 1,\ i = 1\cdots n-1$.. I was able to prove this fact only for $\Omega \subset \mathbb{RP}^2$ in [Craar].

If Ω is an ellipsoid, then obviously $\Lambda = \partial\Omega$ and $\mathcal{A} = \{2\}$. This second property is characteristic of the ellipsoid. (The first one will be treated in section 5.5.)

Corollary 5.4. *Let $\Omega \subset \mathbb{RP}^n$ be a divisible strictly convex set. The closure $\overline{\mathcal{A}}$ of $\mathcal{A}$ has empty interior if and only if Ω is an ellipsoid.*

Proof. Assume Ω is not an ellipsoid. Then, by Theorems 2.4 and 2.5, the set

$$\log\Gamma = \overline{\{[\lambda_0(g) : \cdots : \lambda_n(g)],\ g \in \Gamma\}} \subset \mathbb{RP}^n$$

has nonempty interior.
Now, the set $\mathcal{A}$ contains the vectors $\alpha(g) = (\alpha_i(g)),\ g \in \Gamma$, defined by $\alpha_i(g) = \frac{1-\lambda_n(g)/\lambda_0(g)}{1-\lambda_i(g)/\lambda_0(g)}$. Hence, $\overline{\mathcal{A}}$ contains the image of the well-defined continuous function

$$\begin{array}{ccc} \log\Gamma & \longrightarrow & \mathbb{R}^{n-1} \\ [\lambda_0 : \cdots : \lambda_n] & \longmapsto & \left(\dfrac{1-\lambda_n/\lambda_0}{1-\lambda_1/\lambda_0}, \cdots, \dfrac{1-\lambda_n/\lambda_0}{1-\lambda_{n-1}/\lambda_0}\right). \end{array}$$

This gives the result. □

It is likely that one can replace $\overline{\mathcal{A}}$ by $\mathcal{A}$ in the last proposition. A way to prove that would be to see that the set $\mathcal{A}_{\mathcal{M}}$ defined in section 5.3 contains the interior of $\mathcal{A}$.

5.3. **Ergodic measures.** Let $\Lambda(H\Omega)$ be the set of forward weakly regular points of $H\Omega$, which is obviously Γ-invariant. By Theorem 4.5, the set Λ is given by

$$\Lambda = \{w^+ \in H\Omega,\ w \in \Lambda(H\Omega)\}.$$

Let $\mathcal{M}$ be the set of invariant probability measures of the flow on HM. Each measure $m \in \mathcal{M}$ defines by lifting it a measure $\tilde{m}$ on $H\Omega$ which is invariant under the actions of Γ and the flow. Oseledets' theorem tells us that, for any $m \in \mathcal{M}$, $\Lambda(H\Omega)$ has full $\tilde{m}$-measure, hence Lyapunov exponents are defined $\tilde{m}$-almost everywhere. If m is an ergodic measure, that is invariant sets have zero or full measure, then Lyapunov exponents are constant almost everywhere: to each ergodic measure m we can thus associate its positive Lyapunov exponents $\chi_1(m) \leqslant \cdots \leqslant \chi_{n-1}(m)$.

We can associate, in a one-to-one way, to each invariant probability measure m on HM a Γ-invariant Radon measure $M = M(m)$ on the space of oriented geodesics of Ω given by $\partial^2\Omega = (\partial\Omega \times \partial\Omega) \smallsetminus \Delta$, where $\Delta = \{(x,x),\ x \in \partial\Omega\}$ (see [Kai90] for example). If m is ergodic, Oseledets' theorem implies that for M-almost all $(x,y) \in \partial^2\Omega$, the geodesic from x to y is weakly regular with positive Lyapunov exponents $\chi_1(m) \leqslant \cdots \leqslant \chi_{n-1}(m)$; thus, for M-almost all $(x,y) \in \partial^2\Omega$, the boundary $\partial\Omega$ is approximately $\alpha(m)$-regular at x, with $\alpha(m) = (\alpha_i(m))_{i=1\cdots n-1}$ given by

$$\alpha_i(m) = \frac{2}{\chi_i(m)}.$$

The set $\mathcal{A}_{\mathcal{M}} = \{\alpha(m),\ m \in \mathcal{M}\}$ is an interesting subset of $\mathcal{A}$. As I said before, it might contain the interior of $\overline{\mathcal{A}}$.
The diversity of ergodic measures gives an idea of the complexity of the boundary of a divisible strictly convex set which is not an ellipsoid. Here are some examples.

5.3.1. *Closed orbits.* The easiest examples of ergodic measures are the Lebesgue measures l_g supported by a closed orbit g, associated to a conjugacy class of a hyperbolic element $g \in \Gamma$. The corresponding subset of $\partial^2\Omega$ of full $M(l_g)$-measure is precisely the orbit of (x_g^-, x_g^+) under Γ. This has been treated in the previous part.
Denote by $\mathcal{M}_{Per} = \{l_g,\ g \in \Gamma\}$ the set of ergodic measures supported on closed orbits and define $\mathcal{A}_{\mathcal{M}_{Per}} = \{\alpha(m),\ m \in \mathcal{M}_{Per}\}$. It is a consequence of the Anosov closing lemma that $\mathcal{M}_{Per}$ is dense in the set of ergodic measures, so we could expect the following

Proposition 5.5. *Let $\Omega \subset \mathbb{RP}^n$ be a divisible strictly convex set. The set $\mathcal{A}_{\mathcal{M}_{Per}}$ is dense in $\mathcal{A}_{\mathcal{M}}$.*

Proof. This is a consequence of a nontrivial result one can find in [Kal11] (Theorem 1.4): it states that the vector $\chi(m)$ associated to an ergodic measure can be approximated by a sequence of vectors $(\chi(g_n))$ with $g_n \in \mathcal{M}_{Per}$. The vector $\alpha(m)$ is thus approximated by the sequence $(\alpha(g_n))$. □

5.3.2. *Gibbs measures.* A Gibbs measure is the equilibrium state of a Hölder continuous potential $f : HM \longrightarrow \mathbb{R}$: it is the unique invariant probability measure μ_f such that

$$h_{\mu_f} + \int f \; d\mu_f = \sup\{h_m + \int f \; dm, \; m \in \mathcal{M}\}.$$

The corresponding measure M_f on $\partial^2\Omega$ can always be written as $M_f = FM_f^s \times M_f^u$, where F is a continuous function on $\partial^2\Omega$, and M_f^s and M_f^u are two finite measures on $\partial\Omega$. The three objects are determined by the potential; in particular, M_f^u and M_f^s are given by the Patterson-Sullivan construction, associated to the potentials f and $\sigma * f$, where σ is the flip map, defined on $H\Omega$ by $\sigma(x,[\xi]) = (x,[-\xi])$ (see [Cou03] or [Led95]).
Among Gibbs measures is for instance the Bowen-Margulis measure μ_{BM} which is the measure of maximal entropy of the flow, that is, the equilibrium state associated to the potential $f = 0$. The corresponding measure M_{BM} is given by

$$dM_{BM}(\xi^+,\xi^-) = e^{2\delta(\xi^+|\xi^-)_o} d\mu_o^2(\xi^+,\xi^-),$$

where μ_o is the Patterson-Sullivan measure at an arbitrary point $o \in \Omega$, and $(\xi^+|\xi^-)_o$ is the Gromov product ξ^+ and ξ^- based at the point o: we have $(\xi^+|\xi^-)_o = \frac{1}{2}(b_{\xi^-}(o,x) + b_{\xi^-}(o,x))$ for any point $x \in (\xi^-\xi^+)$ (see [Sul79]).
In [Cra09], I had proved that $\chi^+(\mu_{BM}) = \sum \chi_i(\mu_{BM}) = n-1$. Thus, we get that μ_o-almost every point of $\partial\Omega$ is approximately $\alpha(\mu_{BM})$-regular with $\alpha(\mu_{BM}) = (\alpha_i(\mu_{BM}))_{i=1\cdots n-1}$, such that $(\sum_i 1/\alpha_i(\mu_{BM}))^{-1} = 2(n-1)$. For example, in dimension 2, μ_o-almost every point of $\partial\Omega$ is approximately 2-regular. A question I am not able to answer is to know if, in dimension $n \geqslant 3$, the α_i are all equal to 1 if and only if Ω is an ellipsoid.

5.4. **Shape of the boundary at Lebesgue almost every point.** The Sinai-Ruelle-Bowen (SRB) measure μ^+ is the equilibrium state associated to the potential

$$f^+ = \frac{d}{dt}|_{t=0} \log \det d\varphi^t_{|_{E^u}}.$$

This potential is Hölder continuous because the geodesic flow is $\mathcal{C}^\alpha$ for some $\alpha > 1$. The measure μ^+ is the only measure whose conditional measures $(\mu^+)^u$ along unstable manifolds are absolutely continuous.
Closely related to this measure is the "reverse" SRB measure $\mu^- = \sigma * \mu^+$, which is the equilibrium state of the potential

$$f^- = -\frac{d}{dt}|_{t=0} \log \det d\varphi^t_{|_{E^s}}.$$

The measure μ^- is the only invariant measure whose conditional measures along stable manifolds are absolutely continuous.
In the case of the ellipsoid, μ^+, μ^- and μ_{BM} all coincide, since $f^+ = f^- = 0$, and they are all absolutely continuous; indeed, they coincide with the Liouville measure of the flow. When Ω is not an ellipsoid, the Zariski-density of the cocompact group Γ implies via Livschitz-Sinai theorem that there is no absolutely continuous measure (see [Ben04]). So the three measures are distinct.

The measure μ^+ is also the only one which satisfies the equality in the Ruelle inequality (see [LY85]). Recall that the Ruelle inequality relates the entropy of an invariant measure m to the sum of positive Lyapunov exponents χ^+ of the flow:

$$h_m \leqslant \int \chi^+ \; dm.$$

For example, the topological entropy h_{top} of the flow satisfies

$$h_{top} = h_{\mu_{BM}} \leqslant n-1,$$

with equality if and only if Ω is an ellipsoid (this is the main result of [Cra09]). The measures μ^+ and μ^- have the same entropy h_{SRB} given by

$$h_{SRB} = \int \chi^+ \; d\mu^+ = - \int \chi^- \; d\mu^-,$$

where χ^- is the sum of negative Lyapunov exponents. In particular, if Ω is not an ellipsoid, we have $\int \chi^+ \; d\mu^+ = h_{SRB} < h_{\mu_{BM}} < n-1$. Hence the μ^+-almost sure value $\chi^+(\mu^+)$ of the sum of positive Lyapunov exponents satisfies $\chi^+(\mu^+) < n-1$.
The measure μ^+ corresponds to the measure M^+ on $\partial^2\Omega$ which can be written $M^+ = F^+ M^s \times M^u$, with M^u absolutely continuous, while the measure μ^- corresponds to $M^- = F^- M^u \times M^s$. In particular, we have the following

Proposition 5.6. *Let $\Omega \subset \mathbb{RP}^n$ be a divisible strictly convex set. Then Lebesgue-almost every point of $\partial\Omega$ is approximately α-regular with $\alpha = (\alpha_i)_{i=1\cdots n-1}$ given by*

$$\alpha_i = \frac{2}{\chi_i(\mu^+)}.$$

Since $\partial\Omega$ is also Lebesgue almost-everywhere 2-differentiable by Alexandrov's theorem, we have that $\alpha_i \leqslant 2,\ i = 0 \cdots n-1$. When Ω is an ellipsoid, we have $\alpha_i(SRB) = 2,\ i = 0 \cdots n-1$. Otherwise, the fact that $\chi^+(\mu^+) < 0$ implies that $\chi_1(\mu^+) < 1$ hence $\alpha_1 > 2$. In particular, we recover the fact that the curvature of $\partial\Omega$ is supported on a set of zero Lebesgue-measure.

5.5. **The 2-dimensional case.** In dimension 2, we can understand better the sets Λ and $\mathcal{A}$.

5.5.1. *The set of approximately regular points.* We will see here that the property that $\Lambda = \partial\Omega$ characteristic of the ellipsoid. This is probably true in higher dimensions but we would need a more careful approach.

Proposition 5.7. *Let $\Omega \subset \mathbb{RP}^2$ be a divisible strictly convex set. If Ω is not an ellipse, then there is a point of $\partial\Omega$ at which $\partial\Omega$ is not approximately regular.*

To prove this proposition, we will use the specification property of an Anosov flow, that we recall now (see [KH95]). It roughly means that given a family of pieces of orbits (S below), there exists an orbit that follows these pieces.
A *specification* is a family $S = (S_i)_{i=0\cdots N}$, for some $N \in \mathbb{N} \cup \{+\infty\}$, of pairs $S_i = (w_i, I_i)$ with $w_i \in HM$, $I_i = [t_i, T_i]$, $t_i < T_i$ which satisfy $t_i > T_{i-1}$. For $T > 0$, we say that the specification S is *T-spaced* if $t_i - T_{i-1} \geqslant T,\ i = 1 \cdots N$. Given $\varepsilon > 0$, we say that the orbit of $w \in HM$ *ε-shadows* S if for any $i = 0 \cdots N$, $t \in [t_i, T_i]$, we have $d(\varphi^t(w), \varphi^t(w_i)) \leqslant \varepsilon$.

Theorem 5.8. *The Anosov flow $\varphi^t : HM \longrightarrow HM$ has the specification property: given $\varepsilon > 0$, there exists $T(\varepsilon)$ such that, for any $T(\varepsilon)$-spaced specification S, there exists a point $w \in HM$ whose orbit ε-shadows S.*

We can now give a

Proof of Proposition 5.7. Fix $\varepsilon > 0$, and let $T = T(\varepsilon)$ given by the last theorem.
Choose two periodic points w_1 and w_2 in HM, with distinct positive Lyapunov exponent $\chi_1 < \chi_2$. This is possible if Ω is not an ellipsoid, by Corollary 5.4. For $k \geqslant 0$, let S'_k be the specification

$$S'_k = ((w_1, [0, 2^{2^k}]), (w_2, [T + 2^{2^k}, T + 2^{2^k} + 2^{2^{k+1}}])).$$

If $S = (w_i, [t_i, T_i])_{i=1\cdots N}$ is a specification, we set $\max S = T_N$. For $t \geqslant 0$, we denote by $t + S$ the specification $S = (w_i, [t + t_i, t + T_i])_{i=1\cdots N}$. We set $S_0 = S'_0$, $S_k = T + \max(S_{k-1}) + S'_k,\ k \geqslant 1$.We finally define the infinite specification S by

$$S = (S_0, S_1, \cdots).$$

Since S is T-spaced by construction, there is a point w whose orbit ε-shadows S, and, for $Z \in E^u$, we have

$$|\limsup_{t\to-\infty} \frac{1}{t}\log\|d\varphi^t Z\| - \chi_2| < \eta(\varepsilon),\ \ |\liminf_{t\to+\infty} \frac{1}{t}\log\|d\varphi^t Z\| - \chi_1| < \eta(\varepsilon),$$

with $\lim_{\varepsilon\to 0}\eta(\varepsilon) = 0$. So, if ε is taken so that $\eta(\varepsilon) < (\chi_2-\chi_1)/27$, then w is not forward weakly regular. Theorem 4.5 implies that the boundary $\partial\Omega$ is not approximately regular at the point w^+. □

Proposition 5.7 yields the following

Corollary 5.9. *For any $n \geqslant 2$, there exists a $\mathcal{C}^1$ strictly convex function $f : U \subset \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ which is not approximately regular at some point.*

Notice that is possible to construct by hand a function which is not approximately regular at some point, but this is somehow funny to construct one in this way.

5.5.2. *The range of Lyapunov exponents.* We now turn to the study of $\mathcal{A}$ which benefits from the following observation, which has no equivalent in dimension higher than 2. If $\mu \in \mathcal{M}$ is ergodic, the positive Lyapunov exponent of μ is given by

$$\chi(\mu) = \int \frac{d}{dt}|_{t=0} \log\|d\varphi^t\|\ d\mu,$$

hence the application $\mu \longrightarrow \chi(\mu)$ is continuous. In this case for example, proposition 5.5 is immediate.

Proposition 5.10. *Let $\Omega \subset \mathbb{RP}^2$ be a divisible strictly convex set. Then $\mathcal{A}$ is a closed interval.*

Proof. First, remark that $\mathcal{A}_{\mathcal{M}}$ is the image of the set of ergodic measures by the continuous application

$$\mu \longmapsto \alpha(\mu) = \frac{2}{\int \frac{d}{dt}|_{t=0}\log\|d\varphi^t\|\ d\mu}.$$

As the set of ergodic measures is compact, $\mathcal{A}_{\mathcal{M}}$ is compact.
We now see that $\mathcal{A}_{\mathcal{M}}$ is convex. For that, recall that $\mathcal{A}_{\mathcal{M}_{Per}}$ is dense in $\mathcal{A}_{\mathcal{M}}$. So it suffices to prove that for any $g,\ g' \in \mathcal{M}_{Per}$, $\varepsilon > 0$ and $\lambda \in [0,1]$, we can find $g_\varepsilon \in \mathcal{M}_{Per}$ so that

$$|\chi(g_\varepsilon) - (\lambda\chi(g) + (1-\lambda)\chi(g'))| < \varepsilon.$$

This is a simple application of the shadowing lemma (a particular case of the specification property, see [KH95]).
It remains to see that $\mathcal{A} = \mathcal{A}_{\mathcal{M}}$. Pick a point $w \in \Lambda$ with Lyapunov exponent $\chi(w)$. Consider the measures μ_T defined for $T > 0$ by

$$\int f\ d\mu_T = \frac{1}{T}\int_0^T f(\varphi^t w)\ dt.$$

For $f = \frac{d}{dt}|_{t=0}\log\|d\varphi^t\|$, we have

$$\lim_{T\to+\infty} \frac{1}{T}\int \log\frac{d}{dt}|_{t=0}\|d\varphi^t\|\ d\mu_T = \lim_{T\to+\infty}\frac{1}{T}\log\|d\varphi^T\| = \chi(w).$$

Hence, any accumulation point μ of the family $(\mu_T)_{T>0}$ is an invariant measure such that $\chi(\mu) = \chi(w)$. □

Remark that, in fact, the same proof would prove that

$$\mathcal{A} = \{\limsup_{t\to+\infty}\frac{1}{t}\log\|d_w\varphi\|,\ w \in \Omega\}.$$

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