ENTROPIES OF STRICTLY CONVEX PROJECTIVE MANIFOLDS

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ABSTRACT. Let M be a compact manifold of dimension n with a strictly convex projective structure. We consider the geodesic flow of the Hilbert metric on it which is known to be Anosov. We prove that its topological entropy is less than $n-1$, with equality if and only if the structure is Riemannian hyperbolic. As a corollary, we get that the volume entropy of a divisible strictly convex set is less than $n-1$, with equality if and only if it is an ellipsoid.

1. INTRODUCTION

In 1936, in what seems to be the first general introduction to the notion of locally homogeneous space [21], Charles Ehresmann noticed the following : it is not excluded for the universal covering of some compact locally projective surface to be a bounded convex domain whose boundary would not be analytic. But immediately he added that to his mind such a case should not occur. Thirty years later, Kac and Vinberg [44] proved that this implausible situation was indeed possible.
Such surfaces, and by extension, such manifolds are the main objects of this article. These are compact manifolds which can be written as a quotient Ω/Γ, where Ω is a strictly convex proper open set of the projective space and Γ a subgroup of the projective group acting cocompactly on Ω. Such a manifold is said to be strictly convex projective and Ω is said to be divisible.

Lots of compact manifolds admit strictly convex projective structures. The basic example is a hyperbolic manifold for which the Beltrami-Klein model of hyperbolic space provides such a structure. As observed on many occasions by various authors, for any other strictly convex projective structure, the boundary of the convex set is much less regular. Ehresmann first noticed that it was nowhere analytic. Then Benzécri [11] proved that if the boundary was C^2, then the convex set was an ellipsoid. Finally, from a different point of view, Edith Socié-Methou [42] proved that if the convex set has a C^2 boundary with positive definite Hessian, then, except in the case of an ellipsoid, its group of isometries was compact.
Despite everything, these non-smooth structures are numerous in various senses :

- If a manifold admits a hyperbolic structure then it may also admit some non-smooth strictly convex projective structures ; moreover, the deformation space $\mathcal{G}(M)$ of such structures may be much bigger than the Teichmüller space $\mathcal{T}(M)$ of hyperbolic structures.
  In dimension 2, Goldman [24] proved that it is a real open cell of dimension $16g-16$, where $g \geq 2$ denotes the genus of the surface, whereas $\mathcal{T}(M)$ is only of dimension $6g-6$. In dimension higher than 3, Mostow's rigidity theorem [39] claims that $\mathcal{T}(M)$ is at most a point. It follows from the works of Benzécri [11] and Koszul [33] on affine and projective manifolds that $\mathcal{G}(M)$ is open in the space of projective structures $\mathbb{RP}^n(M)$. In particular, Johnson and Millson [29] constructed non trivial continuous deformations of some hyperbolic structure into strictly convex projective ones.
- There are manifolds which admit strictly convex projective structures but no hyperbolic structure. Such example cannot exist in dimension 2 and 3 but Benoist [10] constructed an example in dimension 4, and Kapovich [30] proved that some Gromov-Thurston manifolds [26] actually provided other examples.

Any strictly convex set Ω carries a Hilbert metric d_Ω (see section 2.1). When Ω is an ellipsoid, (Ω, d_Ω) coincides with the hyperbolic space; in the other cases, the metric is not Riemannian anymore, but comes from a Finsler metric which has the same regularity as the boundary of the convex. Hilbert metric is invariant under any homography, and thus provides a metric on any compact projective manifold $M = \Omega/\Gamma$. With this metric, M is projectively flat : in local projective charts, geodesics, as locally shortest paths, are straight lines.

These structures are for various reasons generalizations of hyperbolic ones. Despite the lack of regularity, we can define a notion of curvature and prove it is constant and strictly negative. Furthermore, Yves Benoist proved the following theorem :

Theorem ([7])**.** *Let Ω be a divisible convex set, divided by Γ. The following statements are equivalent :*

- *the space (Ω, d_Ω) is Gromov-hyperbolic ;*
- *Ω is strictly convex ;*
- *the boundary $\partial\Omega$ of Ω is C^1 ;*
- *Γ is Gromov-hyperbolic.*

This paper can be seen as a continuation of [7], where Benoist initiated the study of the geodesic flow of the Hilbert metric. In particular, Benoist proved similar properties to those of the hyperbolic geodesic flow, namely that the flow was Anosov and topologically mixing. But he already made the following observation, which distinguished the two dynamical systems : whereas hyperbolic geodesic flows admit the Liouville measure as natural invariant measure, the others do not admit any smooth invariant measure.
A major invariant in the theory of dynamical systems (see [31]) is the topological entropy, which roughly speaking measures how the system separates the points, how much it is chaotic. Let us recall briefly its definition. Given a system $\varphi^t : X \longrightarrow X$ on a compact metric space (X, d), we define the distances d_t, $t \geq 0$, on X by $d_t(x, y) = \max_{0\leq s\leq t} \ d(\varphi^s(x), \varphi^s(y)),\ x, y \in X$. The topological entropy of φ is then the well defined quantity

$$h_{top}(\varphi) = \lim_{\epsilon\to 0} \Big[\limsup_{t\to\infty} \frac{1}{t} \log N(\varphi, t, \epsilon) \Big] \in [0, +\infty],$$

where $N(\varphi, t, \epsilon)$ denotes the minimal number of open sets of diameter less than ϵ for d_t needed to cover X.
It is well known that the topological entropy of the hyperbolic geodesic flow is $n-1$ when the manifold is of dimension n. Our main theorem answers a question that emerged during a Finsler meeting at the CIRM in 2005 and provides a new distinction between the Riemannian and the non-Riemannian cases :

Theorem 1.1. *Let φ be the geodesic flow of the Hilbert metric on a strictly convex projective compact manifold M of dimension n. Its topological entropy $h_{top}(\varphi)$ satisfies the inequality*

$$h_{top}(\varphi) \leq (n-1),$$

with equality if and only if the Hilbert metric comes from a Riemannian metric.

The proof of this result is mainly based on results in the Anosov systems theory, developed since the 60's, and on the geometrical approach to second order differential equations made by Patrick Foulon in [22].

Antony Manning [35] noticed that on non positively curved Riemannian manifolds, the topological entropy of the geodesic flow was equal to the volume entropy of the Riemannian metric. The volume entropy of a Riemannian metric g on M measures the exponential asymptotic growth of the volume of balls in the universal covering $\tilde{M}$; it is defined by

$$h_{vol}(g) = \lim_{r\to\infty} \frac{1}{r} \log vol(B(x, r)),$$

where vol denotes the Riemannian volume corresponding to g. We can also consider the volume entropy $h_{vol}(\Omega, d_\Omega)$ of a Hilbert geometry (Ω, d_Ω) and extends the result of Manning in this case. This yields the following rigidity result :

Corollary 1.2. *Let Ω be a strictly proper convex open set in $\mathbb{P}(\mathbb{R}^n)$ divided by a group $\Gamma \in PGL(\mathbb{R}^n)$ such that $M = \Omega\backslash\Gamma$ is compact. Then*

$$h_{vol}(\Omega, d_\Omega) \leq n - 1$$

with equality if and only if Ω is an ellipsoid.

Thus, in the case of a manifold which admits a hyperbolic structure, the maximum of the (topological or volume) entropy characterizes the Teichmüller space $\mathcal{T}(M)$ in $\mathcal{G}(M)$. In any case, we get an entropy function $h : \mathcal{G}(M) \longrightarrow \mathbb{R}$ which takes its values in $(0, n-1]$. That leads to some natural questions:

- what is the infimum of h and is it attained ?
- in the case of a manifold which does not admit any hyperbolic structure, what is the supremum of h and is it attained ?
- how regular is h ?

Let us now explain the contents of the paper.
We begin by some necessary preliminaries consisting of basic facts and notations. We also specify the context of the paper and give some motivations.
We then extend in section 3 the dynamical formalism introduced in [22] to our context. In particular, it allows us to define a notion of curvature in Hilbert geometry, that we prove to be constant and strictly negative, and to make parallel transport along the orbits of the geodesic flow, that will be the main tool of the paper.
In section 4, this parallel transport is related to the action of the geodesic flow, that leads to a new description of the Anosov property. Here the projective flatness of the structures is crucial : working in the universal covering identified with Ω, we can indeed compare the parallel transport with respect to the Hilbert metric with the Euclidean one (section 4.4) ; then an acute study allows us to control the asymptotic behavior of the flow on the tangent space. This part is the technical core of the paper.
Using ergodic properties of hyperbolic systems and some arguments of symmetry, sections 5 and 6 prove the upper bound in theorem 1.1. Motivations and ideas of the proof appear in the preliminaries, section 2.4. These sections also give links between these dynamical properties, namely Lyapunov exponents, the group Γ and the boundary of the convex Ω.
Section 7 explicits the case of equality in theorem 1.1 and provide some complementary facts and considerations about invariant measures. It also gives a large lower bound for the topological entropy in terms of regularity of the boundary of the convex.
Finally, the last section extends the results obtained by Manning, which leads to corollary 1.2.

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2. Preliminaries : concepts and notations

2.1. **Hilbert geometry.**

2.1.1. *Generalities.* Hilbert geometries were introduced by David Hilbert as an example for what is now known as Hilbert's fourth problem : roughly speaking, characterize the metric geometries whose geodesics are straight lines. Hilbert geometries are defined in the following way.
Take a properly convex open set Ω of the projective space $\mathbb{P}^n(\mathbb{R})$, $n \geq 2$, where properly convex means you can find an affine chart in which Ω appears as a relatively compact convex set. The Hilbert metric d_Ω on Ω is defined by

$$d_\Omega(x,y) = \frac{1}{2}|\log([a,b,x,y])|, \ x,y \in \Omega,$$

where a, b are the intersection points of the line (xy) with the boundary $\partial\Omega$ (c.f. Figure 1). $[a,b,x,y]$ denotes the cross ratio of the four points : if we identify the line (xy) with $\mathbb{R} \cup \{\infty\}$, it is defined by $[a,b,x,y] = \frac{|ax|/|bx|}{|ay|/|by|}$.

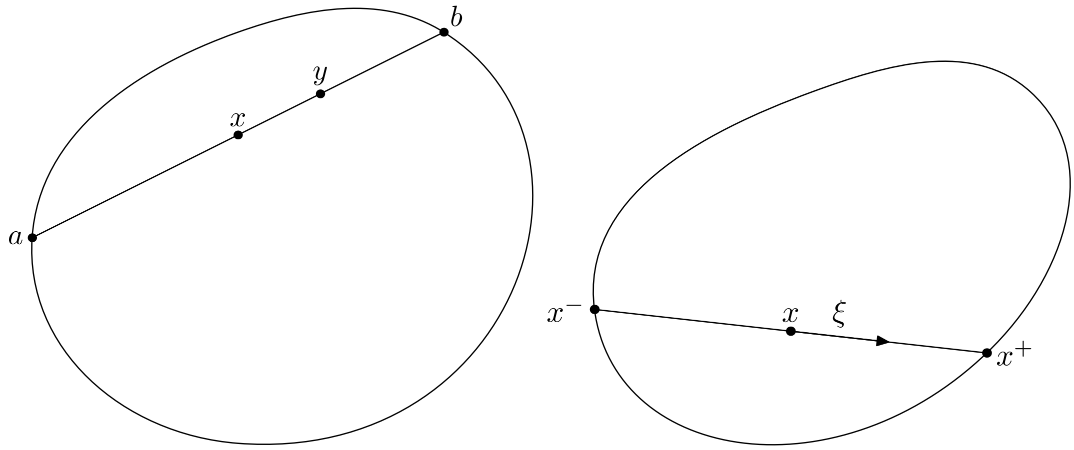

FIGURE 1. The Hilbert distance and the associated Finsler metric

The space (Ω, d_Ω) is then a complete metric space ; see [42] for subsequent details.
In general, the metric is not Riemannian but Finslerian : instead of a quadratic form, we only have a convex norm on each tangent space. Choose an affine chart and a Euclidean metric $|\ .\ |$ on it such that Ω appears as a bounded set of $\mathbb{R}^n$. At the point $x \in \Omega$, this norm of a vector $\xi \in T_x\Omega \setminus \{0\}$ is given by

$$F(x,\xi) = \frac{|\xi|}{2}\left(\frac{1}{|xx^+|} + \frac{1}{|xx^-|}\right), \tag{1}$$

where x^+, x^- are the intersections of the line $\{x + \lambda\xi\}_{\lambda\in\mathbb{R}}$ with the boundary $\partial\Omega$ (see again Figure 1). From this formula, we see that $F : TM\setminus\{0\} \longrightarrow M$ has the same regularity as the boundary $\partial\Omega$. Among all these geometries, those given by ellipsoids are particular : they are the only cases where the metric F is Riemannian (see [42] for more precise statements), and in this case, (Ω, d_Ω) is nothing else than the Klein model for the hyperbolic space. Thus, a relevant problem is to compare the space (Ω, d_Ω) to standard geometries. In particular, note the two following opposite results.

Theorem 2.1.
- [17] *If Ω is C^2 with definite positive Hessian then the metric space (Ω, d_Ω) is bi-Lipschitz equivalent to the hyperbolic space $\mathbb{H}^n$.*
- [18] [13] [43] *(Ω, d_Ω) is bi-Lipschitz equivalent to the Euclidean space if and only if Ω is a convex polytope, that is the convex hull of a finite number of points.*

From a different point of view, Benoist also found general conditions on the boundary for (Ω, d_Ω) to be Gromov-hyperbolic ; see [6].

2.1.2. *Geodesics.* In any case, the space (Ω, d_Ω) is geodesically complete, where by geodesic, we mean a curve which locally minimizes the distance among all piecewise C^1 curves ; indeed, any straight line is a geodesic. The converse is true if and only if the boundary of every plane section of Ω contains at most one open segment (see [42]).

2.1.3. *Isometries.* The subgroup of elements of $PGL(n+1,\mathbb{R})$ which preserve the convex Ω is obviously a subgroup of isometries of the space (Ω, d_Ω). The converse is false in general (see [19]) and there is no known necessary and sufficient condition for this property to be true. The best sufficient condition was given in [19] and specified in [42] : when the space is uniquely geodesic, then $Isom(\Omega, d_\Omega) \subset PGL(n+1,\mathbb{R})$. In particular, this is true when Ω is strictly convex.

2.2. **Hilbert geometry on compact manifolds and divisible convex sets.** We now consider compact manifolds locally modeled on these geometries : we say that a manifold M admits a convex projective structure if there exist a properly convex open set Ω and a subgroup $\Gamma \subset PGL(n+1,\mathbb{R})$ preserving Ω, such that $M = \Omega/\Gamma$. The convex set Ω is then said to be divisible. This structure identifies the universal covering of M with Ω, and its fundamental group $\pi_1(M)$ with Γ. Two structures Ω_1/Γ_1 and Ω_2/Γ_2 are equivalent if there exists $g \in PGL(n+1,\mathbb{R})$ such that $g(\Omega_1) = \Omega_2,\ g\Gamma_1 g^{-1} = \Gamma_2$.
The ellipsoid is once more a particular case of a divisible convex set. As was already noticed by Ehresmann, this is the only analytic model. In fact, for any divisible convex set which is not an ellipsoid, there exists some $0 < \epsilon < 1$ for which $\partial\Omega$ is not $C^{1+\epsilon}$. For more properties, especially about the groups Γ, we refer to the papers of Yves Benoist [7], [5], [8], [9].
Among divisible convex sets, we have to distinguish the strictly convex and the non strictly convex ones ; indeed, if Ω is divisible by a group Γ then the following are equivalent ([7]) :

- the space (Ω, d_Ω) is Gromov-hyperbolic ;
- Ω is strictly convex ;
- the boundary $\partial\Omega$ of Ω is C^1 ;
- Γ is Gromov-hyperbolic.

From these conditions, we see that all convex projective structures on a given manifold M are either all strictly convex or all not strictly convex. In this paper, since we want to study the geodesic flow, we restrict ourselves to manifolds which admit strictly convex projective structures.

2.3. **Geodesic flow.** For every strictly convex projective structure on the compact manifold M, we are able to define the geodesic flow of the Hilbert metric since in this case, there is a unique geodesic between two points, which is a straight line in any projective chart.
In this paper we study the geodesic flow φ^t defined on the homogeneous bundle

$$HM = (TM \setminus \{0\})/\mathbb{R}_+^*,$$

with projection $\pi : HM \longrightarrow M$: a point $w = (x, [\xi]) \in HM$ is given by a point $x \in M$ and a direction $[\xi]$, where $\xi \in TM$. If $w = (x, [\xi]) \in HM$, then its image $\varphi^t(w) = (x_t, [\xi_t])$ is obtained by following the geodesic leaving x in the direction $[\xi]$ during the time t, that is, the length (for the Hilbert metric) of the corresponding geodesic curve between x and x_t is t ; the direction $[\xi_t]$ is the direction tangent to this geodesic at the point x_t.
On the universal covering of M, identified with Ω, the geodesic flow $\tilde{\varphi}^t$ has a very simple interpretation : take a point $x \in \Omega$ and a direction $[\mathbf{x}\mathbf{x}^+]$ for a point $x^+ \in \partial\Omega$; the image $\tilde{\varphi}^t(w)$ of $w = (x, [\mathbf{x}\mathbf{x}^+]) \in H\Omega$ by the geodesic flow is given by $(x_t, [\mathbf{x_t}\mathbf{x}^+])$, where $d_\Omega(x, x_t) = t$. The flow φ^t on HM is then obtained by using the covering map $p : H\Omega \longrightarrow HM$.

The infinitesimal generator of the geodesic flow is a vector field X defined on HM, that is a section $X : HM \longrightarrow THM$ of the tangent bundle of HM. On $H\Omega$, we thus get a Γ-invariant vector field $\tilde{X}$; since orbits of the flow are lines (the metric is said to be flat), once an affine chart and a Euclidean structure on it are fixed, there exists a function $m : HM \longrightarrow \mathbb{R}$ such that

$\tilde{X} = mX^e$, where X^e denotes the infinitesimal generator of the Euclidean metric on Ω. A direct computation gives

$$m(x,[\xi]) = 2\left(\frac{1}{|xx^+|} + \frac{1}{|xx^-|}\right)^{-1} = 2\,\frac{|xx^+|\,|xx^-|}{|x^+x^-|}\ ,$$

so that $F(x,\xi)m(x,[\xi]) = |\xi|$. This property of flatness and the shape of m will be crucial to extend some concepts in section 3 despite the lack of regularity.

The geodesic flow of the Hilbert metric was studied by Yves Benoist, who proved the following

Theorem ([7])**.** *Let $M = \Omega/\Gamma$ be a compact strictly convex projective manifold. Then the geodesic flow of the Hilbert metric on HM is a topologically mixing Anosov flow.*

Recall that a C^1 flow $\varphi^t : W \longrightarrow W$ generated by X on a compact Riemannian manifold W is an Anosov flow if there exist a decomposition

$$TW = \mathbb{R}.X \oplus E^s \oplus E^u,$$

and constants $C, \alpha, \beta > 0$ such that for any $w \in W$ and $t \geq 0$,

$$\|d\varphi^t(Z^s(w))\| \leq Ce^{-\alpha t},\ Z^s(w) \in E^s(w),$$
$$\|d\varphi^{-t}(Z^u(w))\| \leq Ce^{-\beta t},\ Z^u(w) \in E^u(w).$$

Topologically mixing means that for any nonempty open sets $U, V \subset W$, there exists $T \geq 0$ such that for any $t \geq T$, $\varphi^t(U) \cap V \neq \emptyset$.

Such a property was first established by Hadamard [28] in 1898 for the geodesic flow on hyperbolic surfaces, and then generalized to Riemannian manifolds of negative curvature by Anosov in the famous [3]. It is thus a property that is shared by our geometries. Our goal is to study what dynamically separates Riemannian hyperbolic structures from the others; that is to find dynamical properties which characterize hyperbolic metrics among the non Riemannian Hilbert metrics. Benoist made a first step by proving the

Proposition 2.2 ([7], Proposition 6.7)**.** *Let $M = \Omega/\Gamma$ be a compact strictly convex projective manifold. Then the geodesic flow on HM of the Hilbert metric admits no absolutely continuous invariant measure unless the Hilbert metric is Riemannian.*

Recall that a measure μ on a manifold W is said to be absolutely continuous (or smooth) if it is in the Lebesgue class : if A is a Borel subset of W, then $\mu(A) = 0$ as soon as $\lambda(A) = 0$, where λ denotes a Lebesgue measure on HM. The proposition above will be useful in section 7 to determine the case of equality in theorem 1.1.

2.4. **Topological and measure theoretic entropies.** Let $\varphi^t : W \longrightarrow W$ be a flow on a compact manifold W. For $t \geq 0$, we define the distance d_t on W by :

$$d_t(x,y) = \max_{0\leq s\leq t}\ d(\varphi^s(x),\varphi^s(y)),\ \ x,y \in W.$$

For any $\epsilon > 0$ and $t \in \mathbb{R}$, we consider coverings of W by open sets of diameter less than ϵ for the metric d_t. Let $N(\varphi,t,\epsilon)$ be the minimal cardinality of such a covering. The topological entropy ([1]) of the flow is then the (well defined) quantity

$$h_{top}(\varphi) = \lim_{\epsilon\to 0}\left[\limsup_{t\to\infty}\frac{1}{t}\log N(\varphi,t,\epsilon)\right].$$

In a certain sense, it measures how much the system is chaotic. It appears in various and numerous contexts ; the most celebrated result may be this one, essentially due to Margulis (see [37], [32]) : if φ is a topologically mixing Anosov flow, then the number $P_T(\varphi)$ of closed orbits of length less than T satisfies the following asymptotic equivalent, with $h = h_{top}(\varphi)$:

$$P_T(\varphi) \sim \frac{e^{-hT}}{hT}.$$

As an example, the topological entropy of the geodesic flow of a compact hyperbolic manifold of dimension $n \geq 2$ is $(n-1)$. Our main theorem 1.1 states that this property characterizes the hyperbolic structures among all strictly convex projective ones.

To prove this theorem, we will make use of certain objects and results that appear in the ergodic theory of hyperbolic dynamical systems. Here come the motivations for the proof.
Let $\mathcal{M}$ denote the set of φ^t-invariant probability measures. To any $\mu \in \mathcal{M}$ is attached a number h_μ called measure-theoretic entropy ; for definition and basic properties, see [32] or [45]. The variational principle ([25] or [38]) states that

$$h_{top}(\varphi) = \sup_{\mu \in \mathcal{M}} h_\mu,$$

and in the case of a topologically mixing $C^{1+\epsilon}$ Anosov flow (that is relevant for us), we know from Bowen [14] and/or Margulis [36] (see also [32]) that there exists a unique measure μ_{BM}, now known as the Bowen-Margulis measure, such that

$$h_{\mu_{BM}} = h_{top}(\varphi).$$

On a hyperbolic manifold, the Bowen-Margulis measure of the geodesic flow is the natural Liouville measure. From proposition 2.2, we know that in the case of a non Riemannian Hilbert metric, this measure will not be smooth anymore.
Osedelec's theorem [40] and Pesin-Ruelle inequality [41] give a way to calculate $h_{\mu_{BM}}$: if $\mu \in \mathcal{M}$ then the set of regular points is of full measure (see definition 5.1 and theorem 5.2) and

$$h_\mu \leq \int \chi^+ d\mu, \tag{2}$$

where χ^+ is the sum of positive Lyapunov exponents. Proposition 5.3 will give a formula for our Lyapunov exponents which will be sufficient to conclude.

2.5. **Volume entropy of Hilbert geometries.** We define the volume entropy of a Hilbert geometry (Ω, d_Ω), provided it exists, by

$$h_{vol}(\Omega, d_\Omega) = \lim_{r\to\infty} \frac{1}{r} \log vol(B(x,r)). \tag{3}$$

It measures the asymptotical exponential growth of the volume of balls. By volume, we mean the Hausdorff measure associated to the Hilbert metric. Note that, if the convex set is divisible by a group Γ, this volume is Γ-invariant, giving a volume on the manifold Ω/Γ.
The problem of measuring a volume in a Finsler space was already discussed a lot and we will not discuss it again. Look at [16] and [2] for instance.

It is not clear when the limit in (3) exists, but some results are already known : as a consequence of theorem 2.1, if Ω is a polytope then $h_{vol}(\Omega, d_\Omega) = 0$; at the opposite, we have the

Theorem 2.3 ([12]). *If $\partial\Omega$ is $C^{1,1}$, that is with Lipschitz derivative, then $h_{vol}(\Omega, d_\Omega) = n-1$.*

It is conjectured that $h_{vol}(\Omega, d_\Omega) \leq n-1$ for any convex set Ω of dimension n. In [12] the conjecture is proved in dimension $n=2$ and an example is also constructed where $0 < h_{vol} < 1$. Theorem 1.2 will provide numerous examples of convex sets, in any dimension n, whose entropy satisfies

$$0 < h_{vol} < n-1.$$

3. Dynamical formalism

To prove the main theorem, we use the dynamical objects introduced by Patrick Foulon in [22] to study second order differential equations : they provide analogues of Riemannian objects such as covariant differentiation, parallel transport and curvature for any such equation which is regular enough.
We want to apply that formalism to our Hilbert geometries, which are more irregular. The goal of this part is thus to carefully check that these objects are still well defined, and even smooth in some sense, under some specific assumptions. For more details about this, we refer the reader to [22] and to the appendix of [23] for an English version.

3.1. **Directional smoothness.** Assume a smooth vector field X^0 is given on a smooth manifold W. We denote by

- $C_{X^0}(W)$ (or simply C_{X^0}) the set of functions f on W such that, for any $n \geq 0$, $L^n_{X^0} f$ exists;
- $C^p_{X^0}(W)$ (or simply $C^p_{X^0}$) the set of functions $f \in C_{X^0}$ such that, for any $n \geq 0$, $L^n_{X^0} f \in C^p(W)$.

A C_{X^0} (respectively $C^p_{X^0}$) vector field Z will be a section of $W \longrightarrow TW$ which is smooth in the direction X^0, that is, the Lie derivative $L^n_{X^0} Z$ exists (respectively exists and is C^p) for any $n \geq 0$. Equivalently, Z can be locally written as $Z = \sum f_i Z_i$ where the Z_i are smooth vector fields on W, and $f_i \in C_{X^0}$ (respectively $f_i \in C^p_{X^0}$).
When X^0 is a complete vector field, f being in C_{X^0} means that f is smooth all along the orbits of the flow generated by X^0.

Lemma 3.1. *Let $m \in C^1_{X^0}$ and $X = mX^0$. For any C_{X^0} vector field Z,*

(i) *$L_Z m \in C_{X^0}$;*
(ii) *for any $n \geq 0$, the Lie derivative $L^n_X Z = [X[\cdots[X, Z]\cdots]$ is a C_{X^0} vector field.*

In some sense, if $X = mX^0$ with $m \in C^1_{X^0}$,this lemma means that to be smooth with respect to X is equivalent to being smooth with respect to X^0. The proof will make use of the following improved version of Schwartz' theorem.

Lemma 3.2. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a C^1 map. If $\frac{\partial^2 f}{\partial x_i \partial x_j}$ exists and is continuous then so is $\frac{\partial^2 f}{\partial x_j \partial x_i}$ and we have $\frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j}$.*

Proof of the proposition. (i) Let $w \in W$. Since X^0 is smooth, we can find smooth coordinates $(x_0, x_1, \cdots, x_n)$ on a neighbourhood V_w of w such that $X^0 = \frac{\partial}{\partial x_0}$ and $Z = \sum z_i X^i$, where $z_i \in C_X(V_w)$ and $X^i = \frac{\partial}{\partial x_i}$.
Let $f \in C^1_{X^0}$. Then on V_w, we formally have

$$L_{X^0} L_Z f = \sum L_{X^0}(z_i L_{X^i} f) = \sum L_{X^0} z_i L_{X^i} f + \sum z_i L_{X^0}(L_{X^i} f).$$

In fact, this expression makes sense. The first term is well defined and in C_{X^0}. The second one exists from lemma 3.2 ; we even have $L_{X^0} L_{X^i} f = L_{X^i} L_{X^0} f$, so that

$$L_{X^0} L_Z f = L_Z L_{X^0} f + \sum L_{X^0} z_i L_{X^i} f. \tag{4}$$

We now prove that $L^n_{X^0} L_Z m$ exists by induction on n. Assume that for some $n \geq 0$, we know that

$$L^n_{X^0} L_Z m = m_n + L_Z L^n_{X^0} m$$

for some function $m_n \in C_{X^0}$. Then

$$L^{n+1}_{X^0} L_Z m = L_{X^0} m_n + L_{X^0} L_Z (L^n_{X^0} m).$$

But $L^n_{X^0}m \in C^1_{X^0}$, so that we can apply the preceding result (equation (4)) with $f = L^n_{X^0}m$ to get that

$$L_{X^0}L_Z(L^n_{X^0}m) = L_Z L^{n+1}_{X^0}m + g$$

for some function $g \in C_{X^0}$. We thus have

$$L^{n+1}_{X^0}L_Z m = m_{n+1} + L_Z L^{n+1}_{X^0}m$$

with $m_{n+1} = L_{X^0}m_n + g \in C_{X^0}$. That proves the first point.

(ii) The Lie derivative $Z^0_n := L_{X^0}Z$ exists for any $n \geq 0$. Let $Z_0 := Z$ and (formally) $Z_n := L^n_X Z$ for $n \geq 1$. Assume that for some $n \geq 0$, Z_n exists and can be written

$$Z_n = m^n Z^0_n + z_n$$

where z_n is some C_{X^0} vector field. Then

$$\begin{aligned} Z_{n+1} \;=\; [X, Z_n] \;&=\; m[X^0, m^n Z^0_n + z_n] - L_{Z_n}m\; X^0 \\ &=\; m[X^0, z_n] + m^{n+1}Z^0_{n+1} + nm^n L_{X^0}m\; Z^0_{n+1} - L_{Z_n}m\; X^0, \end{aligned}$$

so that

$$Z_{n+1} = m^{n+1}Z^0_{n+1} + z_{n+1}$$

with $z_{n+1} \in C_{X^0}$. That proves the second point. □

3.2. **Foulon's dynamical formalism.** In what follows, M is a smooth manifold and X a complete C^1 second order differential equation on M, that is, a complete C^1 vector field on HM as defined in [22]. We make the assumption that $X = mX^0$ where

- X^0 is a smooth second order differential equation on M ;
- $m \in C^1_{X^0}(HM)$.

Lemma 3.1 claims that to be smooth with respect to either X or X^0 is equivalent, so we will not make the difference between C_X and C_{X^0} functions or vector fields.
We denote by $(\varphi^t)_{t\in\mathbb{R}}$ the flow generated by X. If $w \in HM$, $\varphi.w$ denotes the orbit of w under the flow φ^t, that is, $\varphi.w = \{\varphi^t(w),\ t \in \mathbb{R}\}$. Remark that X and X^0 have the same orbits, up to parametrization. We follow the presentation made in [22].

3.2.1. *The vertical distribution and operator.* The vertical distribution is the smooth distribution $VHM = \ker d\pi$ where $\pi : HM \longrightarrow M$ is the bundle projection. The letter Y will always denote a C_X vertical vector field, and we write $Y \in VHM$. The following lemma is proved in [22] :

Lemma 3.3. *Let $w_0 \in HM$, $Y_1, \cdots, Y_{n-1}$ be vertical vector fields along $\varphi.w_0$ such that, for any $w \in \varphi.w_0$, $Y_1(w), \cdots, Y_{n-1}(w)$ is a basis of V_wHM. Then for any $w \in \varphi.w_0$, the family*

$$X(w), Y_1(w), \cdots, Y_{n-1}(w), [X, Y_1](w), \cdots, [X, Y_{n-1}](w)$$

is a basis of T_wHM.

This lemma allows us to define the vertical operator as the C_X-linear operator such that

$$v_X(X) = v_X(Y) = 0\ ;\ v_X([X, Y]) = -Y.$$

By C_X-linear, we mean that, for any function $f \in C_X$,

$$v_X(fZ) = f v_X(Z).$$

From the very definition, we can check that

$$v_X = m v_{X^0}. \tag{5}$$

3.2.2. *The horizontal operator and distribution.* The horizontal operator $H_X : VHM \longrightarrow THM$ is the C_X-linear operator defined by

$$H_X(Y) = -[X,Y] - \frac{1}{2}v_X([X,[X,Y]]).$$

Lemma 3.1 assures us that this definition makes sense. More precisely, we have

$$[X,Y] = m[X^0,Y] - L_Y m X^0$$

and

$$[X,[X,Y]] = m^2[X^0,[X^0,Y]] + L_X m[X^0,Y] - (L_X L_Y m - mL_{[X,Y]}m)X^0.$$

Since $v_X = mv_{X^0}$, we thus get

$$H_X(Y) = mH_{X^0}(Y) + L_Y m X^0 + \frac{1}{2}L_{X^0}mY. \tag{6}$$

The horizontal distribution $h^X HM$ is defined by

$$h^X HM = H_X(VHM).$$

The verticality lemma 3.3 implies that H_X is injective, so that we get the continuous decomposition

$$THM = \mathbb{R}.X \oplus VHM \oplus h^X HM.$$

By a horizontal vector field $h \in h^X HM$, we will mean a C_X section h of $HM \longrightarrow h^X HM$.

The operators v_X and H_X exchange VHM and $h^X HM$ in the following sense : lemma 3.1 allows us to consider the compositions $v_X \circ H_X$ and $H_X \circ v_X$, and see that for any $Y \in VHM$, $h \in h^X HM$,

$$v_X \circ H_X(Y) = Y,\ H_X \circ v_X(h) = h. \tag{7}$$

In particular, remark that any horizontal vector field h can be written $h = H_X(Y)$, for a unique $Y \in VHM$.

Finally, we can define a pseudo-complex structure

$$J^X : h^X HM \oplus VHM \longrightarrow h^X HM \oplus VHM$$

by setting $J^X = v_X$ on $h^X HM$ and $J^X = -H_X$ on VHM. Equation (7) gives

$$J^X \circ J^X = -Id|_{VHM \oplus h^X HM}.$$

3.2.3. *Projections.* We associate to the decomposition

$$THM = \mathbb{R}.X \oplus VHM \oplus h^X HM$$

the corresponding decomposition of the identity :

$$Id = p^X \oplus p_v^X \oplus p_h^X.$$

We immediately have that

$$p_h^X = H_X \circ v_X. \tag{8}$$

Moreover,

Lemma 3.4. *For any C_X vector field Z, we have*

$$p^X(Z) = p^{X^0}(Z) - L_{v_{X^0}(Z)}(\log m)X^0;$$

$$p_v^X(Z) = p_v^{X^0}(Z) - \frac{1}{2}(L_{X^0}\log m)v_{X^0}(Z);$$

$$p_h^X(Z) = p_h^{X^0}(Z) + (L_{v_{X^0}(Z)}(\log m))X^0 + \frac{1}{2}(L_{X^0}\log m)v_{X^0}(Z).$$

In particular, every projection of Z is still a C_X vector field.

Proof. Let $Z = aX + Y + h = a^0X^0 + Y^0 + h^0$ be the two decompositions of the vector field Z along $\varphi.w$. If we note $y = v_{X^0}(h^0) = v_{X^0}(Z)$, we have using (6)

$$h = H_X(v_X(Z)) = \frac{1}{m}H_X(y) = H_{X^0}(y) + \frac{1}{2m}L_{X^0}m\ y + \frac{1}{m}L_y m\ X^0.$$

Thus

$$h = h^0 + \frac{1}{2}L_{X^0}(\log m)y + L_y(\log m)X^0,$$

and

$$Z = (aX + L_y(\log m)X^0) + (Y + \frac{1}{2}L_{X^0}(\log m)y) + h^0 = a^0X^0 + Y^0 + h^0.$$

Identifying gives the result. □

3.2.4. *Dynamical derivation.* We define an analog of the covariant derivation along X that we call the dynamical derivation and denote by D^X. It is the C_X-differential operator of order 1 defined by

$$D^X(X) = 0,\ D^X(Y) = -\frac{1}{2}v_X([X,[X,Y]]),\ [D^X, H_X] = 0.$$

In our context, being a C_X-differential operator of order 1 means that for any $f \in C_X$,

$$D^X(fZ) = fD^X(Z) + (L_X f)Z.$$

Remark that, on VHM, we can write

$$D^X(Y) = H_X(Y) + [X,Y]. \tag{9}$$

We can also check that

$$D^X = mD^{X^0} + \frac{1}{2}L_{X^0}mId.$$

A vector field Z is said to be parallel along X, or along any orbit, if $D^X(Z) = 0$. This allows us to consider the parallel transport of a C_X vector field along an orbit : given $Z(w) \in TwHM$, the parallel transport of $Z(w)$ along $\varphi.w$ is the parallel vector field Z along $\varphi.w$ whose value at w is $Z(w)$; the parallel transport of $Z(w)$ at $\varphi^t(w)$ is the vector $Z(\varphi^t(w)) = T^t(Z(w)) \in T_wHM$. (See section 4 for more details.) Since D^X commutes with J^X, the parallel transport also commutes with J^X. If X is the generator of a Riemannian geodesic flow, the projection on the base of this transport coincides with the usual parallel transport along geodesics.

3.2.5. *Jacobi endomorphism and curvature.* The Jacobi operator R^X is the C_X-linear operator defined by

$$R^X(X) = 0,\ R^X(Y) = p_v^X([X, H_X(Y)]),\ [R^X, H_X] = 0,$$

R^X is well defined thanks to lemma 3.1 and from lemma 3.4, we get that for any C_X vector field Z, $R^X(Z)$ is also a C_X vector field. Remark that R^X commutes with J^X. On VHM, we have

$$R^X = m^2R^{X^0} + \big(\frac{1}{2}mL^2_{X^0}m - \frac{1}{4}(L_{X^0}m)^2\big)Id. \tag{10}$$

3.3. **Applications to Hilbert geometry.** Let Ω be a strictly convex subset of $\mathbb{RP}^n$ with C^1 boundary. Choose an affine chart and a Euclidean metric on it, such that Ω appears as a bounded set of $\mathbb{R}^n$. On $H\Omega$, we consider the generators $\tilde{X}$ and X^e of the Hilbert and Euclidean geodesic flows. We have $\tilde{X} = mX^e$, with

$$m(w) = 2\ \frac{|xx^+|\ |xx^-|}{|x^+x^-|},\ w = (x, [\xi]).$$

A direct computation gives that, for any $w = (x, [\xi]) \in H\Omega$,

$$L_{X^e}m\ (w) = 2\ \frac{|xx^+| - |xx^-|}{|x^+x^-|}\ ;\ L^2_{X^e}m\ (w) = -\frac{4}{|x^+x^-|},\ L^n_{X^e}m = 0,\ n \geq 3,$$

so that $m \in C^1_{X^e}$. Thus the formalism we introduced in the last section is relevant in this situation, X^e playing the role of X^0.

3.3.1. *Jacobi endomorphism and curvature.* We immediately check that $R^{X^e}=0$. Moreover, we have

Proposition 3.5. *Let Ω be a strictly convex subset of $\mathbb{RP}^n$ with C^1 boundary and $\tilde{X}$ be the generator of the Hilbert metric on Ω. Then*

$$R^{\tilde{X}}|_{VH\Omega\oplus h^{\tilde{X}}H\Omega} = -Id|_{VH\Omega\oplus h^{\tilde{X}}H\Omega}.$$

This proposition means that, in some sense, such Hilbert geometries have constant strictly negative curvature. If the boundary of Ω is C^2, we get that the flag curvature of (Ω, d_Ω) is exactly -1.

Proof. We have

$$\begin{aligned}\frac{1}{2}mL_{X^e}^2m-\frac{1}{4}(L_{X^e}m)^2 &= \frac{1}{2}.\ 2\ \frac{|xx^+||xx^-|}{|x^+x^-|}.\frac{-4}{|x^+x^-|}-\frac{1}{4}.\left(2\ \frac{|xx^+|-|xx^-|}{x^+x^-}\right)^2\\ &= -\frac{4\ |xx^+||xx^-|+(|xx^+|-|xx^-|)^2}{|x^+x^-|^2}=-1.\end{aligned}$$

Using equation (10), we then get $R^{\tilde{X}}|_{VH\Omega\oplus h^{\tilde{X}}H\Omega} = -Id|_{VH\Omega\oplus h^{\tilde{X}}H\Omega}$. □

3.3.2. *The Hilbert form of a Finsler metric.* The vertical derivative of a C^1 Finsler metric F on a manifold M is the 1-form on $TM\setminus\{0\}$ defined for $Z\in T(TM\setminus\{0\})$ by :

$$d_vF(x,\xi)(Z)=\lim_{\epsilon\longrightarrow 0}\frac{F(x,\xi+\epsilon dp(Z))-F(x,\xi)}{\epsilon},$$

where $p: TM \longrightarrow M$ is the bundle projection. This form depends only on the direction $[\xi]$: it is invariant under the Liouville flow generated by the Liouville field $D=\sum \xi_i\frac{\partial}{\partial\xi_i}$. As a consequence, d_vF descends by homogeneity on HM to get a 1-form A called the Hilbert form of F.

Let X be the infinitesimal generator of the geodesic flow of F on HM. Since $[d\pi(X(x,[\xi]))]=[\xi]$, we can define A for any $Z\in THM$ by

$$A(Z)=\lim_{\epsilon\longrightarrow 0}\frac{F(d\pi(X+\epsilon Z))-1}{\epsilon}.$$

Remark that $A(X)=1$ and that $A(Y)=0$ for any vertical vector field.

When X is smooth, the 2-form dA is well defined and we have

$$\imath_X dA=0\ ;\ \ker A = VHM\oplus h^X HM.$$

The following proposition extends this result to some less regular Hilbert geometries.

Proposition 3.6. *Let Ω be a strictly convex subset of $\mathbb{RP}^n$ with C^1 boundary and A the Hilbert form of the Hilbert metric F on Ω. Then*

(i) $\ker A = VH\Omega\oplus h^{\tilde{X}}H\Omega$;
(ii) *A is invariant under the geodesic flow of the Hilbert metric.*

To prove the proposition, we have to make some computations on $H\Omega$, and to make them easier, we will use some special charts, that we introduce now. Choose a point $w=(x,[\xi])\in H\Omega$ with orbit $\tilde{\varphi}.w$. *A chart adapted for this orbit* is an affine chart where the intersection $T_{x^+}\partial\Omega\cap T_{x^-}\partial\Omega$ is contained in the hyperplane at infinity, and a Euclidean structure on it so that the line (xx^+) is orthogonal to $T_{x^+}\partial\Omega$ and $T_{x^-}\partial\Omega$.

All along this paper, when we talk about a **good chart** *or a* **chart adapted** *at $w\in H\Omega$ or its orbit $\tilde{\varphi}.w$, we mean such a chart. (See Figure 2)*

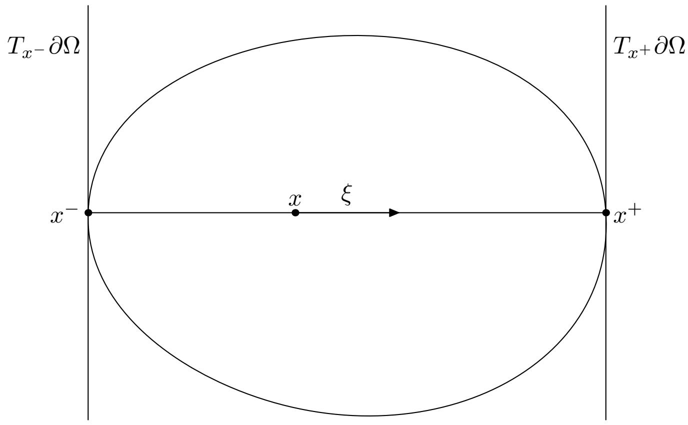

FIGURE 2. A good chart at $w = (x, [\xi])$

In a good chart at w, we clearly have $L_Y m = 0$ along $\tilde{\varphi}.w$ for any $Y \in VH\Omega$. As a corollary of the following proof, we will get that

$$d\pi(V_w H\Omega \oplus h_w^{\tilde{X}} H\Omega) = \left(\mathbf{x}\mathbf{x}^+\right)^{\perp},$$

where orthogonality is taken with respect to the Euclidean metric of the chart.

Proof of proposition 3.6. (i) We only have to prove that $h^{\tilde{X}} H\Omega \subset \ker A$. Let $w_0 = (x_0, [\xi_0])$ be any point in $H\Omega$ and fix a chart for Ω in $\mathbb{R}^n$ which is adapted to w_0, and where $x_0 = 0$ is the origin. Choose a small open neighborhood U of w_0 in $H\Omega$. If U is small enough, we can choose coordinates $(x_1, \cdots, x_n, \xi_2, \cdots, \xi_n)$ on U such that :

- $w_0 = 0$ is the origin ;
- for $w = (x, [\xi]) \in U$, the coordinates $(x_1, \cdots, x_n)$ of x are the Euclidean coordinates in $\mathbb{R}^n$ and $[\xi]$ is identified with the vector

$$\xi = \xi(w) = \frac{\partial}{\partial x_1} + \sum_{i=2}^{n} \xi_i \frac{\partial}{\partial x_i} \in T_x\Omega,$$

where the ξ_i vary in a neighborhood of 0. In other words, $[\xi] = [1 : \xi_2 : \cdots : \xi_n]$, where we make use of homogeneous coordinates on $H_x\mathbb{R}^n$.

We use the associated basis $\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial \xi_j}\right)_{1\leq i\leq n, 2\leq j\leq n}$ on the tangent space $TU \subset TH\Omega$. Remark that all along $\tilde{\varphi}.w_0 \cap U$, we have $\xi = \frac{\partial}{\partial x_1}$.

In this chart, we introduce a new second order differential equation X^0 on U by

$$X^0(w) = X^0(x, [\xi]) = \frac{\partial}{\partial x_1} + \sum_{i=2}^{n} \xi_i \frac{\partial}{\partial x_i}.$$

In particular, we have $X^0(w) = \frac{\partial}{\partial x_1}$ along $\tilde{\varphi}.w_0 \cap U$, and $d\pi(X^0(x, [\xi])) = \xi$ on U. Moreover $\tilde{X}$ can be written as $\tilde{X} = kX^0$, where k is the $C_{\tilde{X}}$ function defined on U by

$$k(w) = \frac{F(d\pi(X^0(w)))}{F(d\pi(\tilde{X}(w)))} = F(x, \xi(w)) = \frac{|\xi(w)|}{m(x, [\xi])},\ w = (x, [\xi]);$$

Along $\tilde{\varphi}.w_0$, we clearly have $L_Y k = 0$.

The vertical distribution on U is given by

$$VU = \mathrm{vect}\{\frac{\partial}{\partial \xi_i}\}_{i\in\{2,\cdots,n\}}.$$

Since $L_Y k = 0$ on $\tilde{\varphi}.w_0$, the pseudo complex structure along $\tilde{\varphi}.w_0$ given by X^0 on $VU \oplus h^{X^0}U$ is very simple : we have

$$\forall j = 2, \cdots, n, \ [X^0, \frac{\partial}{\partial \xi_j}] = -\frac{\partial}{\partial x_j}, \quad [X^0, [X^0, \frac{\partial}{\partial \xi_j}]] = 0,$$

hence

$$\forall j = 2, \cdots, n, \ v_{X^0}(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial \xi_j}, \quad H_{X^0}(\frac{\partial}{\partial \xi_j}) = \frac{\partial}{\partial x_j},$$

thus

$$h^{X^0}U = \mathrm{vect}\{\frac{\partial}{\partial x_i}\}_{i \in \{2, \cdots, n\}}. \tag{11}$$

Equation (6) can be applied with k instead of m. Any horizontal vector field $h \in h^{\tilde{X}}U$ along $\tilde{\varphi}.w_0$ can thus be written

$$h = kH_{X^0}(Y) + \frac{1}{2}(L_{X^0}k)Y, \tag{12}$$

for a certain vector field $Y \in VU$. Since $A(Y) = 0$, we have $A(h) = kA(H_{X^0}(Y))$; so from (11) we only have to prove that for any $i \in \{2, \cdots, n\}$ and $w \in \tilde{\varphi}.w_0$, $A(w)(\frac{\partial}{\partial x_i}) = 0$. But

$$A(\frac{\partial}{\partial x_i}) = \lim_{\epsilon \to 0} \frac{F(d\pi(\tilde{X} + \epsilon \frac{\partial}{\partial x_i})) - 1}{\epsilon} = \lim_{\epsilon \to 0} \frac{F(d\pi(X^0 + \epsilon \frac{\partial}{\partial x_i})) - F(d\pi(X^0))}{\epsilon}$$

so that, for $w \in \tilde{\varphi}.w_0$,

$$A(w)(\frac{\partial}{\partial x_i}) = \lim_{\epsilon \to 0} \frac{F(x, \xi + \epsilon \frac{\partial}{\partial x_i})) - F(x, \xi)}{\epsilon} = D_{(x, \xi(w))}F(\frac{\partial}{\partial x_i}),$$

where we see F as a real valued function on $\Omega \times \mathbb{R}^n \subset \mathbb{R}^{2n}$ with coordinates $(x_1, \cdots, x_n, \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})$. But in our chart, from the formula giving F, we clearly have for any $i \in \{2, \cdots, n\}$, $\frac{\partial}{\partial x_i} \in \ker DF$, which proves that $h^{\tilde{X}}H\Omega \subset \ker A$ along $\tilde{\varphi}.w_0 \cap U$. But all this can be made for any point w_0, so that $h^{\tilde{X}}H\Omega \subset \ker A$ on $H\Omega$.

(ii) Since $A(\tilde{X}) = 1$, to prove that A is invariant under the flow, we only have to prove that its kernel is invariant, which from the first result is equivalent to proving that

$$p^{\tilde{X}}([\tilde{X}, Y]) = p^{\tilde{X}}([\tilde{X}, h]) = 0$$

for any vertical and horizontal vector fields Y and h.

- Since $[\tilde{X}, Y] = -H_{\tilde{X}}(Y) + D^{\tilde{X}}(Y)$, we clearly have $p^{\tilde{X}}([\tilde{X}, Y]) = 0$.
- Now let $w_0 \in H\Omega$ and consider the neighborhood U of w_0 that we've considered before, with its coordinates. Along $\tilde{\varphi}.w_0$, we have $p^{\tilde{X}} = p^{X^0}$, hence

  $$p^{\tilde{X}}([\tilde{X}, h]) = p^{X^0}(k[X^0, h] - L_h k X^0) = kp^{X^0}([X^0, h]) - L_h k.$$

  But, in our chart, we also have $L_h k = 0$ along $\tilde{\varphi}.w_0$: this can be seen directly or using equation (12). Then, if $h = H_{\tilde{X}}(Y)$ and $h^0 = H_{X^0}(Y)$, we have, from (12),

  $$p^{X^0}([X^0, h]) = p^{X^0}([X^0, kh^0 + \frac{1}{2}(L_{X^0}k)Y]) = kp^{X^0}([X^0, h^0]) = 0$$

  on $\tilde{\varphi}.w_0$.

Finally $p^{\tilde{X}}([\tilde{X}, h]) = 0$ on $\tilde{\varphi}.w_0$, and thus on $H\Omega$. □

3.3.3. *Hilbert geometry on manifolds.* Let Ω be a strictly convex subset of $\mathbb{RP}^n$ with C^1 boundary, Γ a discrete subgroup of $Isom(\Omega, d_\Omega)$, without torsion, and $M = \Omega/\Gamma$ the quotient manifold. M inherits a Finsler metric F from the one of Ω. Let X be the generator of the geodesic flow of F on HM.
Of course, we cannot put a Euclidean structure on M, so there exists no X^e on M. But it exists locally, since M is locally isometric to Ω and that's all we need to use Foulon's dynamical formalism on M. We then have a decomposition

$$THM = \mathbb{R}.X \oplus VHM \oplus h^X HM$$

and a pseudo complex structure J^X on $VHM \oplus h^X HM$ that exchanges VHM and $h^X HM$. M has constant strictly negative curvature in the sense that $R^X|_{VH\Omega\oplus h^X H\Omega} = -Id|_{VH\Omega\oplus h^X H\Omega}$.
If A denotes the Hilbert form of F on M, then

$$\ker A = VHM \oplus h^X HM$$

and A is invariant under the geodesic flow.

4. Parallel transport and the Anosov property

4.1. **Action of the flow on the tangent space.** We pick a tangent vector $Z(w) \in T_w HM$. We want to study the behavior of the vector field $Z(\varphi^t(w)) = d\varphi^t(Z(w))$ defined along the orbit $\varphi.w$. Assume

$$Z(w) = Y(w) + h(w) \in V_w HM \oplus h^X_w HM.$$

Since $VHM \oplus h^X HM$ is invariant under the flow, we can write $Z = Y + h$. To find the expressions of Y and h, we write that, since Z is invariant under the flow, the Lie bracket $[X, Z]$ is 0 everywhere on $\varphi.w$.
For that, let $(h_1, \cdots, h_{n-1})$ be a basis of $h^X HM$ of D^X-parallel vectors along $\varphi.w$, that is, $h_i^t = h_i(\varphi^t(w)) = T^t(h_i(w))$, where T^t denotes the parallel transport for D^X and $(h_i(w))_i$ is a basis of $h^X_w HM$. Since D^X and v_X commute, the family $\{Y_i\} = \{v_X(h_i)\}$ is a basis of VHM of D^X-parallel vectors along $\varphi.w$. We have immediately $h_i = H_X(Y_i)$ and

$$[X, Y_i] = -h_i;\ [X, h_i] = -Y_i. \tag{13}$$

Indeed, since Y_i is parallel,

$$[X, Y_i] = D^X(Y_i) - H_X(Y_i) = -h_i.$$

To see the second equality, we write

$$[X, h_i] = p_h^X([X, h_i]) + p_v^X([X, h_i]) + p_X([X, h_i]).$$

But since h_i is parallel, we have

$$p_h^X([X, h_i]) = H_X \circ v_X([X, h_i]) = -H_X \circ v_X([X, [X, Y_i]]) = 2D^X(h_i) = 0,$$

and from the preceding proposition, $p_X([X, h_i]) = 0$; hence

$$[X, h_i] = p_v^X([X, h]) = p_v^X([X, H_X(Y_i)]) = R^X(Y_i) = -Y_i.$$

Then, in this basis, Z can be written as

$$Z = \sum a_i h_i + b_i Y_i,$$

where a_i and b_i are smooth real functions along $\varphi.w$. The formulas (13) give

$$\begin{aligned} [X, Z] = 0 &\iff \textstyle\sum (L_X a_i - b_i) h_i + (L_X b_i - a_i) Y_i = 0 \\ &\iff b_i = L_X a_i;\ a_i = L_X b_i,\ i = 1, \cdots, n-1 \\ &\iff b_i = L_X a_i;\ a_i = L_X^2 a_i,\ i = 1, \cdots, n-1. \end{aligned}$$

From that we get the solution

$$Z(\varphi^t(w)) = d\varphi^t(Z(w)) = \sum A_i e^t (h_i^t + Y_i^t) + B_i e^{-t}(h_i^t - Y_i^t), \tag{14}$$

where

$$A_i = \frac{1}{2}(a_i(w) + b_i(w)),\ B_i = \frac{1}{2}(a_i(w) - b_i(w))$$

depend on the initial coordinates of Z at w.

4.2. **The Anosov property.** Here we give an alternative proof of the Anosov property of the geodesic flow, which was first proved by Yves Benoist in [7]. Our viewpoint shed some new light on the dynamics that will be convenient to get our main theorem 1.1.

Let us define the two diagonals E^u and E^s by

$$E^u = \{Y + H_X(Y), Y \in VHM\}, E^s = \{Y - H_X(Y), Y \in VHM\} = J^X(E^u).$$

We see from (14) that E^u and E^s are invariant under the flow. Furthermore if $Z^s(w) \in E^s(w),\ Z^u(w) \in E^u(w)$, then

$$d\varphi^t(Z^u(w)) = e^t T^t(Z^u(w)),\ d\varphi^t(Z^s(w)) = e^{-t} T^t(Z^s(w)). \tag{15}$$

Theorem 4.1. *The geodesic flow φ^t is an Anosov flow with decomposition*

$$THM = \mathbb{R}.X \oplus E^s \oplus E^u,$$

that is, given a Riemannian metric on HM, there exist constants $C, \alpha, \beta > 0$ such that for any $w \in HM$ and $t \geq 0$,

$$\|d\varphi^t(Z^s(w))\| \leq Ce^{-\alpha t},\ Z^s(w) \in E^s(w),$$

$$\|d\varphi^{-t}(Z^u(w))\| \leq Ce^{-\beta t},\ Z^u(w) \in E^u(w).$$

To prove this theorem, equations (15) above motivate the study of the parallel transport T^t along an orbit : we will thus focus on the exponential behavior of $\|T^t(Z^u(w))\|$ and $\|T^t(Z^s(w))\|$. The proof of the theorem will be completed in section 4.5.

4.3. **Comparison lemma.** Here is the key lemma, due to Yves Benoist [7]. We note $E^{u,s} = E^u \cup E^s$.

Lemma 4.2. *For any Riemannian metric $\|.\|$ on HM, there exists a constant $C > 0$ such that for any $Z(w) \in E^{u,s}(w)$,*

$$C^{-1}\|Z(w)\| \leq F(d\pi(Z(w))) \leq C\|Z(w)\|.$$

Proof. Since $F : TM \to [0, +\infty)$ is a continuous function, so is the function

$$\begin{array}{cccc} F \circ d\pi : & (E^s, \|.\|) & \longrightarrow & [0, +\infty[\\ & u & \longmapsto & F \circ d\pi(u) \end{array}$$

Thus its restriction to the compact $E^s_1 = \{u \in E^s,\ \|u\| = 1\}$ is bounded. Since it is also non zero, there exists $C > 0$ such that, for any $u \in E^s_1$,

$$\frac{1}{C} \leq F(x, d\pi(u)) \leq C,$$

and we conclude the proof using the homogeneity of F. The same works for E^u. □

This lemma gives a way to tackle the problem : we can choose any Riemannian metric on HM, and for any $Z(w) = Y(w) + h(w) \in E^{u,s}(w)$, the exponential behavior of $\|T^t(Z(w))\|$ will be the same as the one of $F(d\pi(T^t(h(w))))$. From now on, we fix a Riemannian metric $\|.\|$ on HM.

4.4. **Parallel transports on $H\Omega$.** We now come back on $H\Omega$ to do some more computations. The Riemannian metric $\|.\|$ and the Finsler metric F on HM give Γ-invariant metrics on $H\Omega$, that we also write $\|.\|$ and F. The lemma 4.2 is still valid.

On $H\Omega$ we work with two vector fields, namely $\tilde{X}$ and X^e, with $\tilde{X} = mX^e$. $\tilde{T}^t$ and T_e^t will denote respectively $D^{\tilde{X}}$ and D^{X^e} parallel transports ; $\tilde{E}^s$, $\tilde{E}^u$ and $\tilde{E}^{u,s} \subset TH\Omega$ correspond to E^s, E^u and $E^{u,s}$.

Lemma 4.3. *If $Y(w) \in V_w H\Omega$ then*

$$\tilde{T}^t(Y(w)) = \left(\frac{m(w)}{m(\varphi^t(w))}\right)^{1/2} T_e^t(Y(w)).$$

Furthermore, in a good chart at w, if $h(w) \in h_w^{\tilde{X}} H\Omega$ then

$$d\pi(\tilde{T}^t(h(w))) = (m(w)m(\varphi^t(w)))^{1/2} d\pi(T_e^t(h(w))).$$

Proof. We look for the unique vector field Y along $\tilde{\varphi}.w$ such that $D^{\tilde{X}}(Y) = 0$ and which takes the value $Y(w)$ at the point w. We recall that

$$D^{\tilde{X}}(Y) = mD^{X^e}(Y) + \frac{1}{2} L_{\tilde{X}}(\log m) Y.$$

Assume we can write $Y = fY^e$, where Y^e is parallel for D^{X^e} along $\tilde{\varphi}.w$. Then f is the solution of the equation

$$L_{\tilde{X}}(\log f) + \frac{1}{2} L_{\tilde{X}}(\log m) = 0,$$

which with $f(w) = 1$ gives

$$f(\tilde{\varphi}^t(w)) = \left(\frac{m(w)}{m(\tilde{\varphi}^t(w))}\right)^{1/2}.$$

Finally,

$$\tilde{T}^t(Y(w)) = \left(\frac{m(w)}{m(\tilde{\varphi}^t(w))}\right)^{1/2} T_e^t(Y(w)). \tag{16}$$

Now, let $h(w) \in h_w^{\tilde{X}} H\Omega$ and h be the parallel transport of h along $\tilde{\varphi}.w$, that is, for $t \in \mathbb{R}$,

$$h(\tilde{\varphi}^t(w)) = T^t(h(w)).$$

Since $J^{\tilde{X}}$ and the parallel transport commute, the vertical vector field $Y = v_X(h)$ defined along $\tilde{\varphi}.w$ is parallel and we have using (9)

$$h = H_{\tilde{X}}(Y) = -[\tilde{X}, Y] + D^{\tilde{X}}(Y) = -[\tilde{X}, Y]$$

along $\tilde{\varphi}.w$. Hence, from (16), we have

$$\begin{aligned}
h = -[\tilde{X}, Y] &= -L_Y m\ X^e - m\ [X^e, Y] \\
&= -L_Y m\ X^e - m\ [X^e, \tfrac{m(w)}{m} Y^e] \\
&= -L_Y m\ X^e - (m(w)m)^{1/2}\ [X^e, Y^e] + m(w)m\ L_{X^e}(m^{-1})\ Y^e \\
&= -L_Y m\ X^e + (m(w)m)^{1/2}\ h^e + m(w)m\ L_{X^e}(m^{-1})\ Y^e.
\end{aligned}$$

In a good chart at w, we have $L_Y m = 0$ on $\tilde{\varphi}.w$, so that :

$$h = (m(w)m)^{1/2}\ h^e + m(w)m\ L_{X^e}(m^{-1})\ Y^e.$$

We then get the result since $d\pi(Y^e) = 0$. $\square$

If f and g are two functions of $t \in \mathbb{R}$, $f(t) \asymp g(t)$ will mean that $f(t) = O(g(t))$ and $g(t) = O(f(t))$, that is there exists $C > 0$ such that $C^{-1}|f(t)| \leq |g(t)| \leq C|f(t)|$ for t large enough.

The following proposition gives a link between the parallel transport and the boundary of Ω. It will be useful in the next section.

Proposition 4.4. *Let $Z(w) \in \tilde{E}^{u,s}(w)$. In a good chart at w,*

$$\|\tilde{T}^t(Z(w))\| \asymp \left(\frac{|x_t x^+|^{1/2}}{|x_t y_t^+|} + \frac{|x_t x^+|^{1/2}}{|x_t y_t^-|} \right),$$

where $x_t = \pi(\tilde{\varphi}^t(w))$ and $y_t^\pm$ are the intersections of the line $\{x_t + \lambda d\pi(\tilde{T}^t Z(w))\}_{\lambda \in \mathbb{R}}$ with the boundary $\partial\Omega$. (c.f. Figure 3)

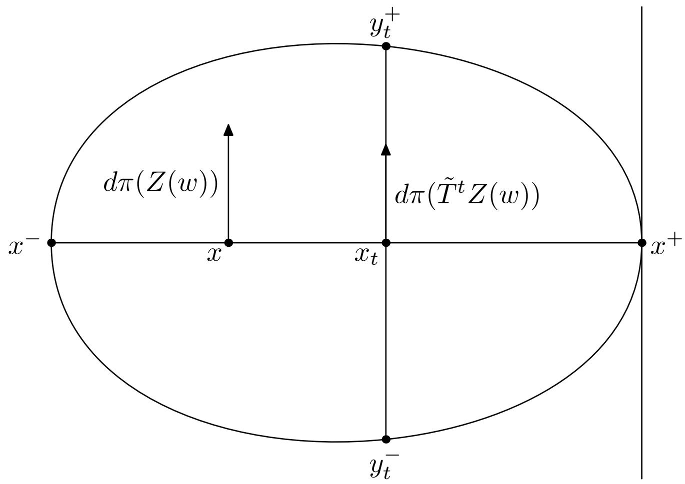

FIGURE 3. Parallel transport on $H\Omega$

Proof. Let choose a good chart at w. We have for $Z(w) = Y(w) + h(w) \in \tilde{E}^{u,s}(w)$,

$$\|\tilde{T}^t(Z(w))\| \asymp F(d\pi(\tilde{T}^t(h(w))) \asymp |x_t x^+|^{1/2} F(d\pi(T_e^t(h(w))))$$

from lemma 4.3. But

$$F(d\pi(T_e^t h(w))) = |d\pi(T_e^t h(w))| m^{-1}(d\pi(T_e^t h(w))) = |d\pi(h(w))| m^{-1}(d\pi(T_e^t h(w))).$$

Since

$$m^{-1}(d\pi(T_e^t h(w))) = \frac{1}{2}\left(\frac{1}{|x_t y_t^+|} + \frac{1}{|x_t y_t^-|} \right),$$

we get

$$\|\tilde{T}^t(Z(w))\| \asymp \left(\frac{|x_t x^+|^{1/2}}{|x_t y_t^+|} + \frac{|x_t x^+|^{1/2}}{|x_t y_t^-|} \right).$$

□

4.5. **Proof of the Anosov property.**

Lemma 4.5. *In a good chart at $w = (x, [\xi])$ we have*

$$\frac{|x_t x^-|}{|x_t x^+|} = e^{2t} \frac{|x x^-|}{|x x^+|}.$$

In particular the following asymptotic expansion holds :

$$|x_t x^+| = \frac{|xx^+|^2}{m(w)} e^{-2t} + O(e^{-4t}).$$

Proof. From the fact that $d_\Omega(x, x_t) = t$, a direct calculation yields

$$xx_t = \frac{e^{2t} - 1}{\frac{1}{|xx^-|} + \frac{1}{|xx^+|} e^{2t}},$$

thus

$$\frac{|x_t x^-|}{|x_t x^+|} = \frac{|xx^-| + |x_t x|}{|xx^+| - |x_t x|} = \frac{|xx^-| + \frac{e^{2t}-1}{\frac{1}{|xx^-|} + \frac{1}{|xx^+|}e^{2t}}}{|xx^+| - \frac{e^{2t}-1}{\frac{1}{|xx^-|} + \frac{1}{|xx^+|}e^{2t}}} = \frac{1 + \frac{|xx^-|}{|xx^+|}}{1 + \frac{|xx^+|}{|xx^-|}} e^{2t} = e^{2t} \frac{|xx^-|}{|xx^+|}.$$

□

We can now follow Benoist's ideas to get theorem 4.1.

Proof of theorem 4.1. For $v \in E^s$, we know from lemma 4.2 that there exists $C > 0$ such that

$$\frac{\|d\varphi^t(v)\|}{\|v\|} \leq C^2 \frac{F(d\pi(d\varphi^t(v))))}{F(d\pi(v))}.$$

Let $E_1^s = \{v \in E^s, \|v\| = 1\}$ be the set of unit "stable" vectors and

$$f : E_1^s \times \mathbb{R} \longrightarrow \mathbb{R}$$

the continuous function defined by

$$f(v, t) = \frac{F(d\pi(d\varphi^t(v)))}{F(d\pi(v))} = \frac{F(d\pi(T^t(v)))}{F(d\pi(v))} e^{-t}.$$

- We first show that, for any $v \in E_1^s$, the function $f(v, .)$ is a strictly decreasing bijection from $[0, +\infty)$ to $(0, 1]$.

  Indeed, let $v \in E_1^s$ and $\tilde{v} \in \tilde{E}_1^s(w) \subset T_w H\Omega$ such that $dp(\tilde{v}) = v$, where $p : H\Omega \to HM$ is the covering map. Choose a chart adapted to $w = (x, [\xi])$. In that chart, the vector $d\pi(\tilde{T}^t \tilde{v})$ is orthogonal to $\mathbf{x_t x^+}$ with respect to the Euclidean structure on the chart; hence so are $\mathbf{x_t y_t^+}$ and $\mathbf{x_t y_t^-}$. Lemma 4.3 gives

$$\begin{aligned} F(d\pi(T^t(v))) = F(d\pi(\tilde{T}^t(\tilde{v}))) &= (m(w) m(\tilde{\varphi}^t(w)))^{1/2} F(d\pi(T_e^t(h))) \\ &= (m(w) m(\tilde{\varphi}^t(w)))^{1/2} \frac{|d\pi(h)|}{m(d\pi(T_e^t(h)))}, \end{aligned}$$

  where h denotes the horizontal component of $\tilde{v}$. Since $F(d\pi(v)) = \dfrac{|d\pi(h)|}{m(d\pi(h))}$, we get

$$f(v, t) = \frac{m(w)^{1/2}}{m(d\pi(h))} e^{-t} \frac{(m(\tilde{\varphi}^t(w)))^{1/2}}{m(d\pi(T_e^t(h)))}.$$

  But

$$\begin{aligned} e^{-t} \frac{(m(\tilde{\varphi}^t(w)))^{1/2}}{m(d\pi(T_e^t(h)))} &= e^{-t} \left(2 \frac{|x_t x^+||x_t x^-|}{|x^- x^+|} \right)^{1/2} \frac{1}{2} \left(\frac{1}{|x_t y_t^+|} + \frac{1}{|x_t y_t^-|} \right) \\ &= \frac{1}{\sqrt{2}} e^{-t} \left(\frac{1}{|x^- x^+|} \frac{|x_t x^-|}{|x_t x^+|} \right)^{1/2} \left(\frac{|x_t x^+|}{|x_t y_t^+|} + \frac{|x_t x^+|}{|x_t y_t^-|} \right). \end{aligned}$$

Finally, from lemma 4.5, we get

$$f(v,t) = \frac{1}{\sqrt{2}} \frac{m(w)^{1/2}}{m(d\pi(h))} \left(\frac{|xx^-|}{|x^-x^+||xx^+|}\right)^{1/2} \left(\frac{|x_tx^+|}{|x_ty_t^+|} + \frac{|x_tx^+|}{|x_ty_t^-|}\right) = \frac{|xx^-|}{|x^-x^+|m(d\pi(h))} \left(\frac{|x_tx^+|}{|x_ty_t^+|} + \frac{|x_tx^+|}{|x_ty_t^-|}\right).$$

The strict convexity of Ω implies that the function $h : t \mapsto \frac{|x_tx^+|}{|x_ty_t^+|} + \frac{|x_tx^+|}{|x_ty_t^-|}$ is strictly decreasing on $[0,+\infty)$ and the C^1 regularity of $\partial\Omega$ that $\lim_{t\to+\infty} h(t) = 0$. Thus, the same holds for $f(v,.)$.

- We now copy Benoist's proof. Choose $0 < a < 1$. From the first point, for any $v \in E_1^s$ there is a unique time $T_a(v)$ such that $f(v,T_a(v)) = a$, that defines a continuous function $T_a : E_1^s \to \mathbb{R}$. Since E_1^s is compact, this function is bounded by some $t_a > 0$, such that
$$\forall t \geq t_a,\ \forall v \in E_1^s,\ f(v,t) \leq a.$$
Now, remark that for any $v \in E_1^s$ and $t,s \geq 0$, $f(v,t+s) = f(v,t)f(d\varphi^t(v),s)$. Thus we get, for t large enough and any $v \in E_1^s$,
$$f(v,t) \leq af(d\varphi^{t-t_a}(v), t-t_a) \leq \cdots \leq a^{[t/t_a]}f(d\varphi^{t-[t/t_a]t_a}(v), t-[t/t_a]t_a) \leq M_a e^{-\alpha t},$$
with $M_a = \max\{f(v,t),\ 0 \leq t \leq t_a, v \in E_1^s\} < +\infty$ and $\alpha = -\log(a)/t_a > 0$.
That means that for any $v \in E^s$,
$$\|d\varphi^t(v)\| \leq C^2 M_a e^{-\alpha t}\|v\|.$$

Reversing the time and using J^X, we get the result for $v \in E^u$, which completes the proof. □

5. Lyapunov exponents

5.1. Generalities.

Definition 5.1. *Let $\varphi = (\varphi^t)$ be a C^1 flow on a Riemannian manifold W. The point $w \in W$ (or its orbit $\varphi.w$) is regular if there exists a φ^t-invariant decomposition*
$$TW = \mathbb{R}.X + \oplus_{i=1}^p E_i$$
along $\varphi.w$ and real numbers
$$\chi_1(w) < \cdots < \chi_p(w),$$
called Lyapunov exponents, such that, for any vector $v_i \in E_i\backslash\{0\}$,
$$\lim_{t\to\pm\infty} \frac{1}{t} \log\|d\varphi^t(v_i)\| = \chi_i(w),$$
and
$$\lim_{t\to\pm\infty} \frac{1}{t} \log|detd\varphi^t| = \sum_{i=1}^p \chi_i(w)\dim E_i.$$

The essential result is the following

Theorem 5.2 (Osedelec's ergodic multiplicative theorem, [40])**.** *Let $\varphi = (\varphi^t)$ be a C^1 flow on a compact Riemannian manifold W. For any φ^t-invariant measure, the set Λ of regular points is of full measure.*

Let us come back to our case, and pick a regular point $w \in \Lambda \subset HM$. Obviously, the Lyapunov decomposition in definition 5.1 will be a subdecomposition of the Anosov decomposition, that is
$$THM = R.X \oplus E^s \oplus E^u = R.X \oplus (\oplus_{i=1}^p E_i^s) \oplus (\oplus_{j=1}^q E_j^u).$$
Since φ^t is an Anosov flow, the Lyapunov exponents are nonzero ; the positive Lyapunov exponents χ_i^+ will come from the unstable distribution and the negative χ_i^- from the stable. The

following proposition relates the Lyapunov exponents and the parallel transport. Together with proposition 4.4, we get a link between the Lyapunov exponents and the shape of the boundary $\partial\Omega$.

Proposition 5.3. *Let $w \in \Lambda$ be a regular point. The Lyapunov decomposition is given by*

$$THM = R.X \oplus \left(\oplus_{i=1}^{p}(E_i^s \oplus E_i^u)\right),$$

with $E_i^s = J^X(E_i^u)$. Furthermore, the corresponding Lyapunov exponents are given by

$$\chi_i^{\pm}(w) = \pm 1 + \eta_i(w)$$

where

$$-1 < \eta_1(w) < \cdots < \eta_p(w) < 1$$

are the Lyapunov exponents of the parallel transport T^t at w.

Proof. Choose $Z_i^u(w) \in E_i^u(w)$ corresponding to the Lyapunov exponent $\chi_i^+(w)$. Then, from equations (15),

$$\chi_i^+(w) = \chi(w, Z_i^u(w)) = \lim_{t\to\infty} \frac{1}{t} \log \|d\varphi^t(Z_i^u(w))\| = 1 + \lim_{t\to\infty} \frac{1}{t} \log \|T^t(Z_i^u(w))\| = 1 + \eta_i(w).$$

Let us now consider the corresponding stable vector $Z_i^s(w) = J^X(Z_i^u(w)) \in E_i^s(w)$. We can write $Z_i^u(w) = h(w) + Y(w)$, but then $Z_i^s(w) = -h(w) + Y(w)$ and $d\pi(T^t(Z_i^s(w))) = -d\pi(T^t(Z_i^u(w)))$. Hence

$$\|T^t(Z_i^s(w))\| \asymp F(d\pi(T^t(Z_i^s(w)))) = F(d\pi(T^t(Z_i^u(w)))) \asymp \|T^t(Z_i^u(w))\|,$$

which gives

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{t} \log \|d\varphi^t(Z_i^s(w))\| &= \lim_{t\to\infty} \frac{1}{t} \log(e^{-t}\|d\varphi^t(Z_i^s(w))\|) \\ &= -1 + \lim_{t\to\infty} \frac{1}{t} \log \|T^t(Z_i^u(w))\| = -1 + \eta_i(w). \end{aligned}$$

Thus, $\chi_i^{\pm}(w) = \pm 1 + \eta_i(w)$. Finally, since the Lyapunov exponents are nonzero, we have $-1 < \eta_1(w) < \cdots < \eta_p(w) < 1$.

□

5.2. **Shape of the boundary.** Here we specify the relation between the Lyapunov exponents and the boundary $\partial\Omega$. For this we come back to the function

$$g(t, Z(w)) = \left(\frac{|x_t x^+|^{1/2}}{|x_t y_t^+|} + \frac{|x_t x^+|^{1/2}}{|x_t y_t^-|} \right),$$

which appears in the proposition 4.4.

We know from [7] that, in our context of a divisible strictly convex set, the metric space (Ω, d_Ω) is Gromov-hyperbolic. Then proposition 1.8 of [6] tells us that

$$|x_t y_t^+| \asymp |x_t y_t^-|$$

since the points y_t^+, x^-, y_t^-, x^+ is a harmonic "quadruplet" (see [6] for (here not relevant) details). Thus,

$$g(t, Z(w)) \asymp \frac{|x_t x^+|^{1/2}}{|x_t y_t^+|}. \tag{17}$$

Assume w is a regular point, choose $Z_i(w) \in \tilde{E}_i^{u,s}(w)$ and look at the asymptotic exponential behavior of the function $g(t, Z_i(w))$: we have

$$\lim_{t\to\infty} \frac{1}{t} \log g(t, Z_i(w)) = \eta_i(w),$$

that is, for any $\epsilon > 0$ and t large enough,

$$e^{(\eta_i(w)-\epsilon)t} \leq g(t) \leq e^{(\eta_i(w)+\epsilon)t} \tag{18}$$

for t large enough.

What does this mean on the boundary ? Let x^+, x^- be two distinct points on $\partial\Omega$. Choose an affine chart and a Euclidean metric such that $T_{x^+}\partial\Omega$ and $T_{x^-}\partial\Omega$ are parallel and $|x^+x^-| = 1$: we can thus identify a point $x \in (x^+x^-)$ as a real in $(0,1)$ with $x^+ = 0,\ x^- = 1$.
Given a vector $v \in T_{x^+}\partial\Omega$, we look at the section of Ω by the plane vect$\{v, \mathbf{x}^+\mathbf{x}^-\}$, and call $y^\pm(v,x)$ the distance from x to the boundary points $y^\pm(x)$, intersections of $\partial\Omega$ and the line $\{x \pm \lambda v\}_{\lambda>0}$ (see figure 4).
We have the following

Proposition 5.4. *Assume the line (x^+x^-) is the projection of a regular orbit of the flow, with Lyapunov exponents $\chi_i^\pm = \pm 1 + \eta_i,\ i = 1 \cdots p$. Then there exists a decomposition of the tangent space*

$$T_{x^+}\partial\Omega = \oplus_i^p H_i(x^+),$$

and constants C_ϵ for any $\epsilon > 0$, such that, if $v_i \in H_i(x^+)$, then

$$C_\epsilon^{-1}\ x^{(1+\eta_i+\epsilon)/2} \leq y^\pm(v_i, x) \leq C_\epsilon\ x^{(1+\eta_i-\epsilon)/2}$$

for small x.

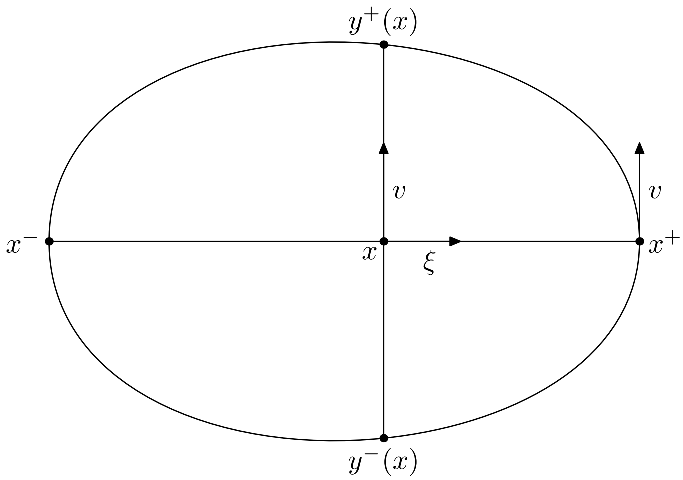

Figure 4.

Proof. We first use lemma 4.5 with $w = (x, [\xi])$, x being the middle point of the segment $[x^+x^-]$ and $\xi = \mathbf{x}\mathbf{x}^+$. We have thus

$$|xx^+| = |xx^-| = m(w) = \frac{1}{2},$$

which gives

$$x_t = |x_t x^+| = \frac{1}{2}e^{-2t}(1 + o(1)).$$

Hence

$$t = \log(x_t^{-1/2}) + O(1). \tag{19}$$

Write $F_i = d\pi(\tilde{E}_i^s)$, $H_i(x^+) = x^+ + F_i$, and pick $v_i \in H_i(x^+)$. Note $y_t^\pm = y^\pm(v_i, x_t)$. From (17) and (18), there exists $0 < C < 1$ such that

$$C^{-1}e^{(\eta_i(w)-\epsilon)t}x_t^{-1/2} \leq \frac{1}{y_t^\pm} \leq Ce^{(\eta_i(w)+\epsilon)t}x_t^{-1/2};$$

hence, using (19),

$$D^{-1}x_t^{-(\eta_i(w)+1)/2+\epsilon} \leq y_t^\pm \leq Dx_t^{-(\eta_i(w)+1)/2-\epsilon},$$

for a constant $0 < D < 1$.

□

Remark that when Ω is an ellipsoid, every point is regular and all the η_i are 0 ; -1 and 1 are the only Lyapunov exponents. In the next section, we see that if Ω is not an ellipsoid, then the Lyapunov exponents vary from a point to another.

5.3. **Lyapunov exponents of a periodic orbit.** Every periodic orbit on HM corresponds to a conjugacy class $[\gamma]$ in the group Γ. As we know from [7], every such element is biproximal, that is : if $(\lambda_i)_{1\leq i\leq n}$ are its (non-necessary distinct) eigenvalues ordered as $|\lambda_1| \geq |\lambda_2| \cdots \geq |\lambda_{n+1}|$, then $|\lambda_1| > |\lambda_2|$ and $|\lambda_{n+1}| < |\lambda_n|$. The length of this periodic orbit is given by

$$l_\gamma = \frac{1}{2}(\log|\lambda_1| - \log|\lambda_{n+1}|).$$

Let us do the study in dimension 2. Take an element $\gamma \in \Gamma$ conjugated to the matrix

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \in SL_3(\mathbb{R})$$

with $\lambda_i \in \mathbb{R}$, $|\lambda_1| > |\lambda_2| > |\lambda_3|$. The line $(\gamma^-\gamma^+)$ is its axis and γ^0 its third fixed point. We look at the picture in the chart given by the plane $\{x_1 + x_3 = 0\} \subset \mathbb{R}^3$, with the following coordinates:

$$\gamma^- = [0:0:1],\ \gamma^+ = [1:0:0],\ \gamma^0 = [0:1:0].$$

This is a good chart for the periodic orbit we are looking at. Choose a point $x \in (\gamma^-\gamma^+)$ with coordinates $[a_0 : 0 : 1 - a_0]$ where $a_0 \in (0,1)$ and let $w = (x, [\gamma^-\gamma^+])$. The point $x_n = \gamma^n.x$ is given by

$$x_n = [a_n : 0 : 1 - a_n],$$

with

$$a_{n+1} = \frac{\lambda_1 a_n}{\lambda_1 a_n + \lambda_2(1-a_n)}.$$

Now, we look at a vector $v = \mathbf{xm} \in \gamma^-\gamma^{+\perp}$ with $m = [a_0 : b_0 : 1 - a_0]$, $b_0 \in \mathbb{R}$. Let $m_n = \gamma^n.m = [a_n : b_n : 1 - a_n]$, $v_n = \mathbf{x_n m_n}$, so that $|v_n| = |b_n|$. Then (b_n) is given by

$$b_{n+1} = \frac{\lambda_2 b_n}{\lambda_1 a_n + \lambda_2(1-a_n)} = \frac{\lambda_2}{\lambda_1}\frac{a_{n+1}}{a_n} b_n,$$

which leads to

$$b_n = \left(\frac{\lambda_2}{\lambda_1}\right)^n \frac{b_0}{a_0} a_n.$$

Since $\lim_{n\to\infty} a_n = 1$, we get

$$b_n \asymp \left(\frac{\lambda_2}{\lambda_1}\right)^n.$$

Let $Z(w) \in T_w H\Omega$ such that $d\pi(Z(w)) = v$. Since γ is an isometry for F, we have, with the notations of proposition 4.4,

$$\begin{aligned} 1 \asymp F(x,v) = F(x_n, v_n) \quad &\asymp \quad \left|\frac{\lambda_2}{\lambda_1}\right|^n \frac{1}{|x_n\gamma^+|^{1/2}} \left(\frac{|x_n\gamma^+|^{1/2}}{|x_n y_n^+|} + \frac{|x_n\gamma^+|^{1/2}}{|x_n y_n^-|} \right) \\ &\asymp \quad \left|\frac{\lambda_2}{\lambda_1}\right|^n e^{nl_\gamma} \|T^{nl_\gamma}(Z(w))\|, \end{aligned}$$

by using lemma 4.5. Thus

$$\|T^{nl_\gamma}(Z(w))\| \asymp \left|\frac{\lambda_1}{\lambda_2}\right|^n e^{-nl_\gamma}$$

and

$$\lim_{t\to+\infty}\frac{1}{t}\log\|T^t(Z(w))\| = \lim_{n\to\infty}\frac{1}{nl_\gamma}\log\|T^{nl_\gamma}(Z(w))\| = -1+2\,\frac{\log|\lambda_1/\lambda_2|}{\log|\lambda_1/\lambda_3|}.$$

All this can be generalized to any dimension by sectioning the convex set, so that we get the following result.

Proposition 5.5. *The Lyapunov exponents $(\eta_i(\gamma))$ of the parallel transport along a periodic orbit corresponding to $\gamma \in \Gamma$ are given by*

$$\eta_i(\gamma) = -1+2\,\frac{\log\lambda_0-\log\lambda_i}{\log\lambda_0-\log\lambda_{p+1}},\ \ i=1\cdots p,$$

where $\lambda_0 > \lambda_1 > \cdots > \lambda_p > \lambda_{p+1}$ denote the moduli of the eigenvalues of γ. The corresponding Lyapunov exponents are given by

$$\chi_i^+(\gamma) = 2\,\frac{\log\lambda_0-\log\lambda_i}{\log\lambda_0-\log\lambda_{p+1}},\ \ i=1\cdots p,$$

$$\chi_i^-(\gamma) = -2+2\,\frac{\log\lambda_0-\log\lambda_i}{\log\lambda_0-\log\lambda_{p+1}},\ \ i=1\cdots p.$$

This result was already known by Yves Benoist [7], but stated in another form and context ; he used it to prove that the geodesic flow is topologically mixing, and to prove proposition 2.2. Remark that in the case of a hyperbolic structure, we have $p=1$ and $\lambda_1=1$, so that $\eta_1=0$. In fact, we can find a Riemannian metric $\|.\|$ on HM for which the parallel transport is an isometry. In the other cases, the proposition proves that it is not possible anymore.

6. Symmetric considerations

In that section we prove the upper bound in the main theorem 1.1. We already know that the Lyapunov exponents can be written

$$\chi_i^\pm = \pm1+\eta_i,\ \ i=1\cdots p.$$

Thus

$$\chi^+ = \sum_{i=1}^{p}\dim E_i\ \chi_i^+ = (n-1)+\eta,$$

where $\eta=\sum_{i=1}^p \dim E_i\ \eta_i$, so that we get from Ruelle inequality

$$h_{top} \leq (n-1)+\int_{HM}\eta\ d\mu_{BM}.$$

We aim to prove that

$$\int_{HM}\eta\ d\mu_{BM}=0.$$

Since the measure μ is ergodic and η is φ^t-invariant, this is equivalent to the fact that $\eta=0$ almost everywhere for μ_{BM}. However, as we saw in the section 5.3, η is not identically 0 on Λ unless Ω is an ellipsoid.

The main remark is that the Hilbert metric is a reversible Finsler metric. The flows φ^t and φ^{-t} are thus conjugated by the flip map

$$\begin{array}{rccc}\sigma: & HM & \longrightarrow & HM\\ & w=(x,[\xi]) & \longmapsto & (x,[-\xi]).\end{array}$$

which is a C^∞ involutive diffeomorphism.

We say that

- a subset A of HM is symmetric if $\sigma A = A$;

- a function $f : A \to \mathbb{R}$ defined on a symmetric set A is symmetric if $f \circ \sigma = f$, antisymmetric if $f \circ \sigma = -f$.
- a measure μ on HM is symmetric if $\sigma * \mu = \mu$.

Lemma 6.1. (i) *The application σ exchanges the stable and unstable foliations.*

(ii) *The set Λ of regular points is a symmetric set and $d\sigma$ preserves the Lyapunov decomposition by sending $E_i^s(w)$ to $E_i^u(\sigma(w))$, for any $w \in \Lambda$.*

(iii) *The function $\eta : \Lambda \longrightarrow \mathbb{R}$ is antisymmetric.*

(iv) *The Bowen-Margulis measure μ_{BM} of φ^t is symmetric.*

Proof. (i) is well known.
(ii) If $w \in \Lambda$, then from the very definition 5.1 of a regular point,

$$\lim_{t\to\infty} \frac{1}{t} \log \|d_w \varphi^{-t}(Z(w))\| = -\lim_{t\to\infty} \frac{1}{t} \log \|d_w \varphi^{t}(Z(w))\| = -\chi(w, Z(w)),$$

for $Z(w) \in T_w HM$. Since $\varphi^{-t} = \sigma \circ \varphi^t \circ \sigma$, we thus have

$$-\chi(w, Z(w)) = \lim_{t\to\infty} \frac{1}{t} \log \|d_w \varphi^{-t}(Z(w))\| = \lim_{t\to\infty} \frac{1}{t} \log \|d_{\sigma(w)} \varphi^{t}(d_w\sigma(Z(w)))\| = \chi(\sigma(w), d_w\sigma(Z(w))),$$

which proves that $\sigma(w)$ is also regular, hence Λ is symmetric. We also get the decomposition

$$T_{\sigma(w)}HM = R.X(\sigma(w)) \oplus (\oplus_i^p (E_i^s(\sigma(w)) \oplus E_i^u(\sigma(w))))$$

with

$$E_i^s(\sigma(w)) = d\sigma(E_i^u(w)),\ E_i^u(\sigma(w)) = d\sigma(E_i^s(w)).$$

(iii) We then have

$$\chi_i^+(\sigma(w)) = -\chi_{p+1-i}^-(w), \tag{20}$$

so that

$$\eta_i(\sigma(w)) = -\eta_{p+1-i}(w).$$

We finally get

$$\eta(\sigma(w)) = \sum_{i=1}^{p} \dim E_i(\sigma(w))\ \eta_i(\sigma(w)) = -\eta(w).$$

(iv) Since φ^t and φ^{-t} are conjugated by σ, $\sigma * \mu_{BM}$ and μ_{BM} are invariant measures of φ^t and φ^{-t} and they have the same entropy. Hence $\sigma * \mu_{BM} = \mu_{BM}$ by unicity of the measure of maximal entropy. □

This lemma gives the first part of theorem 1.1, that is

Proposition 6.2. *Let φ be the geodesic flow on the Hilbert metric on a compact strictly convex projective manifold M of dimension n. Its topological entropy $h_{top}(\varphi)$ satisfies the inequality*

$$h_{top}(\varphi) \leq (n-1).$$

Proof. Since μ_{BM} is symmetric and η antisymmetric, we have $\int \eta\ d\mu_{BM} = 0$, which yields

$$h_{top}(\varphi) = h_{\mu_{BM}} \leq (n-1).$$

□

7. Invariant measures and the equality case

7.1. **The equality case.** Here we deal with the equality case in theorem 1.1. This is closely related to the equality case in the Ruelle inequality (2), that is : for which measures $\mu \in \mathcal{M}$ do we have

$$h_\mu = \int \chi^+ \, d\mu \ ?$$

Ledrappier and Young answered this question in the first part of [34] :

Theorem 7.1 ([34], Theorem A)**.** *Let $\varphi^t : W \longrightarrow W$ be a $C^{1+\epsilon}$ flow on a compact manifold W. Then an invariant measure μ has absolutely continuous conditional measures on unstable manifolds if and only if*

$$h_\mu = \int_W \chi^+ \, d\mu.$$

(In the original paper, this is proved for C^2 diffeomorphisms, but it extends to our case. See [4] for a complete presentation.)
From this theorem we can now prove the

Proposition 7.2. $h_{top} = n-1$ *if and only if the Hilbert metric is Riemannian.*

Proof. $h_{top} = n-1$ if and only if $h_{\mu_{BM}} = n-1$, that is the Bowen-Margulis measure satisfies the equality in the Ruelle inequality. But from theorem 7.1, it is equivalent to the absolute continuity of its conditional measures on unstable manifolds, that is the absolute continuity of the Margulis measures μ^u on strong unstable manifolds : recall that Bowen-Margulis measure was constructed by Margulis as a local product $\mu^s \times \mu^u \times dt$, where the measures μ^s and μ^u were measures on strong stable and strong unstable manifolds with adequate properties (see [36], [37] or [32] for more details). It follows from the symmetry of this construction that μ^u is absolutely continuous if and only if μ^s is so, that is if and only if μ_{BM} is absolutely continuous. The proposition 2.2 concludes the proof. □

We can add some remarks to this proof and connect it with some well known results in the ergodic theory of hyperbolic systems. In our context of a topologically mixing Anosov flow, we indeed know from [15] that there exists only one invariant measure μ^+, called the Sinai-Ruelle-Bowen (SRB) measure, which satisfies the equality in (2). This measure is ergodic and characterized by any of the following equivalent facts :

- μ^+ satisfies the equality in (2) ;
- the conditional measures $(\mu^+)^u$ on unstable manifolds is smooth ;
- the equality

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(\varphi^t(x)) \, dt = \int f(x) \, d\mu^+(x), \tag{21}$$

  holds for λ-almost every point $x \in HM$.

Reversing the time, we get the SRB measure μ^- for φ^{-t}, which is equal to μ^+ if and only if one of the two measures is smooth. Roughly speaking, those two measures are the smoothest invariant measures of the system.
In the case of a hyperbolic geodesic flow, the Bowen-Margulis and the SRB measures (for φ^t and φ^{-t}) coincide with the Liouville measure. This is not true anymore when the Hilbert metric is not Riemannian: both SRB measures are not smooth anymore and we then get three measures which are of interest, each one being singular with respect to the others. The two measures μ^+ and μ^- are related via σ by $\mu^+ = \sigma * \mu^-$; hence σ is a smooth diffeomorphism of HM which sends the measure μ^- to the measure μ^+ which is singular with respect to μ^-.

The function η is invariant under the flow and thus is constant almost everywhere with respect to either of the ergodic measures μ_{BM}, μ^- and μ^+. We have already seen that η was zero μ_{BM}-almost everywhere. But with respect to μ^- or μ^+, η is equal to a constant $\eta_{SRB} < 0$ almost everywhere, since we can prove that $h_{\mu^+} = h_{\mu^-}$ and use $h_{\mu^+} = n - 1 + \eta_{SRB} < n - 1$.

7.2. **A large lower bound for the entropy.** We conclude this section by giving a lower bound for the topological entropy in terms of regularity of the boundary. When Ω is not an ellipsoid, then the boundary is known to be C^α for a certain $\alpha > 1$ but the supremum α_Ω of such α's is stricly less than 2. Equivalently (see [7]), the boundary is β-convex, for a certain $\beta > 2$, that is there exists a constant $C > 0$, such that, for any $p,\ p' \in \partial\Omega$,

$$d_{\mathbb{R}^n}(p', T_p\partial\Omega) \geq C|pp'|^\beta,$$

where $d_{\mathbb{R}^n}$ denotes the Euclidean distance. It was proved by Guichard [27] that the corresponding infimum $\beta_\Omega > 2$ satisfies

$$\frac{1}{\beta_\Omega} + \frac{1}{\alpha_\Omega} = 1.$$

Proposition 7.3. *Let $M = \Omega/\Gamma$, where Ω is not an ellipsoid, and assume $\partial\Omega$ is β-convex for a $\beta \in (2, +\infty)$. Then*

$$h_{top}(\varphi) > \frac{2}{\beta}(n-1).$$

Proof. The β-convexity of the boundary implies there exists $C > 0$ such that, for any $t \geq 0$,

$$|x_t x^+| \geq C|y_t y_t^+|^\beta.$$

Hence

$$\frac{|x_t x^+|^{1/2}}{|y_t y_t^+|} \geq D|x_t x^+|^{1/2-1/\beta},$$

for a certain constant $D > 0$. Thus any positive Lyapunov exponent χ_i^+ satisfies

$$\chi_i^+ = 1 + \eta_i \geq 1 + \lim_{t\to\infty} \frac{1}{t} \log |x_t x^+|^{1/2-1/\beta} = \frac{2}{\beta},$$

from proposition 4.4 and lemma 4.5. Finally, since μ^+ satisfies the Ruelle entropy formula (2), we have

$$h_{top}(\varphi) > h_{\mu^+} \geq \frac{2}{\beta}(n-1).$$

□

8. Volume entropy

On the universal covering $\tilde{M}$ of a compact Riemannian manifold (M, g), we can consider the volume entropy $h_{vol}(g)$ of $(\tilde{M}, g)$, which measures the asymptotic exponential growth of volume of balls in $\tilde{M}$:

$$h_{vol}(g) = \lim_{r\to\infty} \frac{1}{r} \log vol(B(x, r)),$$

where *vol* denotes the Riemannian volume corresponding to g. In [35], Anthony Manning proved the following result :

Theorem 8.1. *Let h_{top} be the topological entropy of the geodesic flow of g on HM. We always have*

$$h_{top} \geq h_{vol}(g).$$

Furthermore, if the sectional curvature of M is < 0 then

$$h_{top} = h_{vol}(g).$$

In his PhD thesis, Daniel Egloff [20] extends this result for some regular Finsler manifolds. Let us check that Manning's proof still works in the special case we are dealing with here.

Proposition 8.2. *Let $\varphi^t : HM \longrightarrow HM$ be the geodesic flow of the Hilbert metric on the strictly convex projective manifold $M = \Omega/\Gamma$ and h_{top} denote his topological entropy. Then*

$$h_{top} = h_{vol}(d_\Omega).$$

The proof is similar to the one by Manning and we do not reproduce it here. The only point we have to check is the following technical lemma that Manning proved using negative curvature. Here we can compute it directly.

Lemma 8.3. *The distance between corresponding points of two geodesics $\sigma, \tau : [0, r] \to \Omega$ is at most $d_\Omega(\sigma(0), \tau(0)) + d_\Omega(\sigma(r), \tau(r))$.*

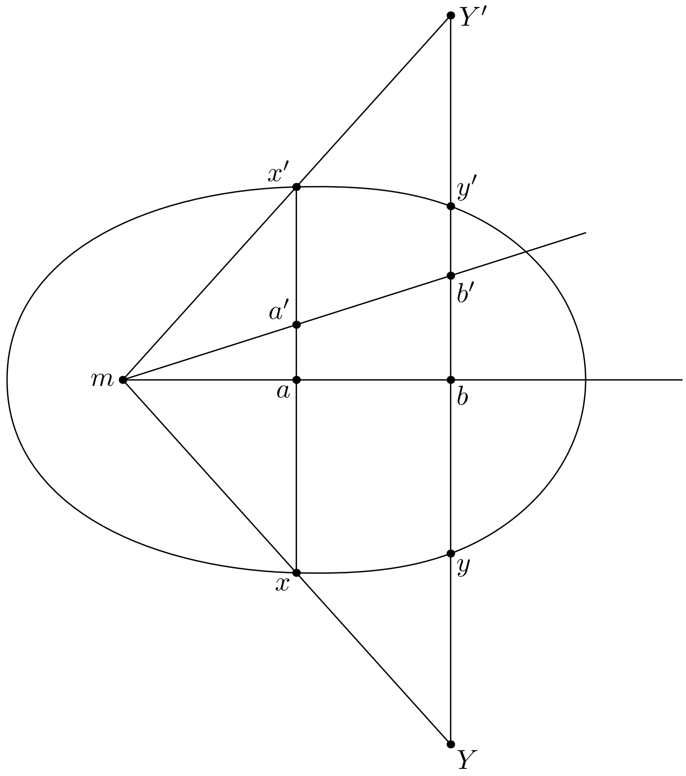

FIGURE 5.

Proof. There are two cases : either σ and τ meet each other or not. Anyway, by joining the point $\sigma(0)$ and $\tau(r)$ with a third geodesic, we see we only have to prove that the distance between two different lines going away from the same point (but not necessary with the same speed) increases.
So suppose $c, c' : \mathbb{R} \to \Omega$ are two lines beginning at the same point $m = c(0) = c'(0)$. Take two pairs of corresponding points $(a, a') = (c(t_1), c'(t_1)), (b, b') = (c(t_2), c'(t_2))$ with $t_2 > t_1 \geq 0$. We want to prove that $d_\Omega(a, a') < d_\Omega(b, b')$. As it is obvious if $t_1 = 0$, assume $t_1 > 0$ and note x, x' and y, y' the points on the boundary $\partial\Omega$ of Ω such that x, a, a', x' and y, b, b', y' are on the same line, in this order. Note also $Y = (mx) \cap (bb')$ and $Y = (mx') \cap (bb')$, so that by convexity of Ω, the six points Y, y, b, b', y', Y' are different and on the same line, in this order. The two lines (aa') and (bb') meet at a certain point that we can send at infinity by an homography. So we can assume the two lines are parallel (c.f. figure 5).
Then it follows from Thales' theorem that

$$1 > [x, a, a', x] = [Y, b, b', Y'] > [y, b, b', y'],$$

so that

$$d_\Omega(a, a') = |\log([x, a, a', x])| < |\log([y, b, b', y'])| = d_\Omega(b, b').$$

□

As a corollary of this proposition and theorem 1.1, we get corollary 1.2.

References

[1] R. L. Adler and M. H. McAndrew A. G. Konheim. Topological entropy. *Trans. Amer. Math. Soc.*, 114:309–311, 1965.

[2] J. C. Álvarez Paiva and A. C. Thompson. Volumes on normed and Finsler spaces. In *A sampler of Riemann-Finsler geometry*, volume 50 of *Math. Sci. Res. Inst. Publ.*, pages 1–48. Cambridge Univ. Press, 2004.

[3] D. V. Anosov. Geodesic flows on closed Riemannian manifolds with negative curvature. *Proc. Inst. Steklov*, 90:1–235, 1967.

[4] L. Barreira and Y. B. Pesin. *Nonuniform hyperbolicity.* Cambridge university Press, 2007.

[5] Y. Benoist. Convexes divisibles 2. *Duke Math. Journ.*, 120:97–120, 2003.

[6] Y. Benoist. Convexes hyperboliques et fonctions quasisymétriques. *Publ. Math. IHES*, 97:181–237, 2003.

[7] Y. Benoist. Convexes divisibles 1. *Algebraic groups and arithmetic, Tata Inst. Fund. Res. Stud. Math.*, 17:339–374, 2004.

[8] Y. Benoist. Convexes divisibles 3. *Annales Scientifiques de l'ENS*, 38:793–832, 2005.

[9] Y. Benoist. Convexes divisibles 4. *Invent. Math.*, 164:249–278, 2006.

[10] Y. Benoist. Convexes hyperboliques et quasiisométries. *Geom. Dedicata*, 122:109–134, 2006.

[11] Jean-Paul Benzécri. Sur les variétés localement affines et localement projectives. *Bull. Soc. Math. France*, 88:229–332, 1960.

[12] G. Berck, A. Bernig, and C. Vernicos. Volume entropy of Hilbert geometries. To appear in Pacific Journal of Mathematics.

[13] A. Bernig. Hilbert geometry of polytopes. *Archiv der Mathematik*, 92:314–324, 2009.

[14] R. Bowen. The equidistribution of closed geodesics. *Amer. J. Math*, 94:413–423, 1972.

[15] R. Bowen and D. Ruelle. The ergodic theory of axiom A flows. *Inventiones Math.*, 29:181–202, 1975.

[16] D. Burago, Y. Burago, and S. Ivanov. *A Course in Metric Geometry.* American Mathematical Society, 2001.

[17] B. Colbois and P.Verovic. Hilbert geometry for strictly convex domains. *Geometriae Dedicata*, 105:29–42, 2004.

[18] B. Colbois, C. Vernicos, and P. Verovic. Hilbert geometry for convex polygonal domains. Preprint, 2008.

[19] P. de la Harpe. On Hilbert's metric for simplices. In *Geometric group theory, Vol. 1*, volume 181 of *London Math. Soc. Lecture Note Ser.*, pages 97–119. Cambridge Univ. Press, 1993.

[20] D. Egloff. *Some developments in Finsler Geometry.* PhD thesis, Université de Fribourg, 1995.

[21] C. Ehresmann. Espaces localement homogènes. *L'ens. Math.*, 35:317–333, 1936.

[22] P. Foulon. Géométrie des équations différentielles du second ordre. *Ann. Inst. Henri Poincaré*, 45:1–28, 1986.

[23] P. Foulon. Estimation de l'entropie des systèmes lagrangiens sans points conjugués. *Ann. Inst. H. Poincaré Phys. Théor.*, 57(2):117–146, 1992. With an appendix, "About Finsler geometry", in English.

[24] W. M. Goldman. Convex real projective structures on compact surfaces. *J. Diff. Geom.*, 31:791–845, 1990.

[25] T. N. T. Goodman. Relating topological entropy and measure entropy. *Bull. of the London Mathematical Society*, 3:176–180, 1971.

[26] M. Gromov and W. Thurston. Pinching constants for hyperbolic manifolds. *Invent. Math.*, 89(1):1–12, 1987.

[27] O. Guichard. Régularité des convexes divisibles. *Ergodic Theory Dynam. Systems*, 25(6):1857–1880, 2005.

[28] J. Hadamard. Les surfaces à courbures opposées et leurs lignes géodésiques. *J. Mathématiques Pures et Appliquées*, 4:27–73, 1898.

[29] D. Johnson and J. J. Millson. Deformation spaces associated to compact hyperbolic manifolds. In *Discrete groups in geometry and analysis, 1984)*, volume 67 of *Progr. Math.*, pages 48–106. Birkhäuser Boston, 1987.

[30] M. Kapovich. Convex projective structures on Gromov-Thurston manifolds. *Geom. Topol.*, 11:1777–1830, 2007.

[31] A. Katok. Fifty years of entropy in dynamics : 1958-2007. *Journal of Modern Dynamics*, 1:545–596, 2007.

[32] A. Katok and B. Hasselblatt. *Introduction to the modern theory of dynamical systems.* Cambridge Univ. Press, 1995.

[33] J.-L. Koszul. Déformations de connexions localement plates. *Ann. Inst. Fourier (Grenoble)*, 18(fasc. 1):103–114, 1968.

[34] F. Ledrappier and L.-S. Young. The metric entropy of diffeomorphisms. *Ann. of Math.*, 122:509–574, 1985.

[35] A. Manning. Topological entropy for geodesic flows. *Ann. of Math.*, 110:567–573, 1979.

[36] G. Margulis. Certain measures associated with y-flows on compact manifolds. *Functional Analysis and Its Applications*, 4:55–67, 1969.

[37] G. A. Margulis. *On Some Aspects of the Theory of Anosov Systems.* Springer Monographs in Mathematics, 2004.

[38] M. Misiurewicz. A short proof of the variational principle for a $\mathbb{Z}_+^n$ action on a compact space. *Astérisque*, 40:147–187, 1976.

[39] G. D. Mostow. Quasi-conformal mappings in n-space and the rigidity of the hyperbolic space forms. *Publ. Math. IHES*, 34:53–104, 1968.
[40] V. I. Osedelec. A multiplicative ergodic theorem. *Trans. Moscow Math. Soc.*, 19:197–231, 1968.
[41] D. Ruelle. An inequality for the entropy of differentiable maps. *Bol. Soc. Bras. de Mat.*, 9:83–87, 1978.
[42] E. Socié-Méthou. *Comportements asymptotiques et rigidité en géométrie de Hilbert.* PhD thesis, Université de Strasbourg, 2000. http://www-irma.u-strasbg.fr/annexes/publications/pdf/00044.pdf.
[43] C. Vernicos. Lipschitz characterisation of convex polytopal Hilbert geometries. Preprint, 2008.
[44] È. B. Vinberg and V. G. Kac. Quasi-homogeneous cones. *Mat. Zametki*, 1:347–354, 1967.
[45] P. Walters. *An introduction to ergodic theory.* Springer-Verlag New York, 1982.

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