The geodesic flow of Finsler and Hilbert geometries

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Abstract. This is a survey of the dynamics of the geodesic flow of Hilbert geometries. The main idea is to compare this flow with the geodesic flow of negatively curved Finsler or Riemannian manifolds, by making links between various useful objects and by comparing results and questions.

Contents

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1 Introduction

The main character of this survey is the geodesic flow of Hilbert geometries. What is quite well understood is its behaviour on a compact quotient manifold of a Hilbert geometry defined by a strictly convex set: the geometry has then some flavour of negatively curved spaces and the flow inherits strong hyperbolicity properties. We should keep this example in mind as a motivation when reading this chapter.
In most of the chapter, we will be interested in those Hilbert geometries which enjoy some kind of hyperbolicity properties. They are defined by strictly convex open sets with $\mathcal{C}^1$ boundary. Indeed, we would like to use techniques and concepts inspired by hyperbolic geometry. Recall that the Hilbert geometry defined by the ellipsoid is the Riemannian hyperbolic space.

Hilbert geometries are examples of Finsler geometries. We will constantly try to give connections between Hilbert geometries and the Riemannian and Finsler negatively curved worlds, where lots of objects and ideas are similar. Analogies will be made, differences will be pointed out, either on tools or on problems and results.
Some proofs are provided when they enlighten the purpose or when they are not fully available in the literature. Other proofs are omitted or briefly sketched and references provided.

The contents of this survey explain better how we will talk about the geodesic flow of Hilbert geometries.
Section 2 is dedicated to basic notions about Finsler and Hilbert geometries, and also about their geodesic flow. We also collect some facts concerning individual isometries and subgroups of isometries of Hilbert geometries.
In Section 3, we introduce differential objects which appear to be useful in the study of Finsler geodesic flows. These objects are fairly general and describe connections between the geometry and the dynamics of the geodesic flow. Most of them come from Riemannian geometry and we explain how they can fit into Finsler and Hilbert geometry.

In sections 4 and 5 we study the geodesic flow of the Hilbert geometries defined by a strictly convex set with $\mathcal{C}^1$ boundary. The goal of Section 4 is to describe stable and unstable bundles and manifolds and relate them with the previous objects. In Section 5, we make a more careful study at the infinitesimal level. In particular, we relate the asymptotic exponential behaviour along an orbit to the shape of the boundary $\partial\Omega$ at the extremal point of the orbit.

I would like to stress that, until this point, no assumption had been made on the existence of quotients. The previous parts provide thus general tools which may be useful in the study of any quotient space. They are illustrated by the last three parts where we study global properties of the geodesic flow of a compact quotient manifold of a strictly convex Hilbert geometry. Among these quotients, compact Riemannian hyperbolic manifolds play a very special role and then can be characterized by various geometrical or dynamical properties.

In Section 7 we prove that the geodesic flow of a compact quotient of a strictly convex Hilbert geometry has the Anosov property. This has consequences on the regularity of the boundary of the geometry. We give a first rigidity result involving the regularity of the boundary.
In Section 8 we get interested in the ergodic theory of the geodesic flow. We recall some concepts and well known results and questions in Riemannian geometry and explains then their counterparts, as well as some results, in Hilbert and Finsler geometry.
Finally, in Section 9, we study various notions of entropies involved in Hilbert geometry. They allow us to make other connections between geometry and dynamics.

2 Preliminaries and notations

2.1 Hilbert geometries

A *Hilbert geometry* is a metric space (Ω, d_Ω) where

- Ω is a *properly convex open set* of the real projective space $\mathbb{RP}^n$, $n \geqslant 2$; *properly* means there exists a projective hyperplane which does not intersect the closure of Ω, or, equivalently, there is an affine chart in which Ω appears as a relatively compact set;

- d_Ω is the distance on Ω defined, for two distinct points x, y, by

$$d_\Omega(x, y) = \frac{1}{2}\log[a, b, x, y],$$

  where a and b are the intersection points of the line (xy) with the boundary $\partial\Omega$ (chosen as in Figure 1) and $[a, b, x, y]$ denotes the cross ratio of the four points : if we identify the line (xy) with $\mathbb{R} \cup \{\infty\}$, it is defined by $[a, b, x, y] = \frac{|ay|/|by|}{|ax|/|bx|}$.

Figure 1. The Hilbert distance

2.2 (Regular) Finsler metrics and their geodesic flow

Definition 2.1. A *Finsler metric* on a manifold M is a field of (non-necessarily symmetric) norms on M, that is a function $F : TM \longrightarrow [0, +\infty)$ such that:

- $F(x, \lambda u) = \lambda F(x, u)$, $(x, u) \in TM$, $\lambda \geqslant 0$;
- $F(x, u + v) \leqslant F(x, u) + F(x, v)$, $x \in M$, $u, v \in T_x M$.

A Finsler metric defines a (non-symmetric) distance d_F on M: the distance between two points x and y of M is the minimal Finsler length $L_F(c)$ of a $\mathcal{C}^1$ curve $c : [0, 1] \longrightarrow M$ from x to y, that is,

$$d_F(x, y) = \inf_c L_F(c) = \inf_c \int_0^1 F(\dot{c}(t))\ dt.$$

We will say that a Finsler metric F is *regular* if F is $\mathcal{C}^2$ and the boundary of its unit balls $B(x, 1) = \{u \in T_x M,\ F(x, u) < 1\}$, $x \in M$ has positive definite Hessian (for some (hence any) fixed Euclidean metric on $T_x M$). This is the minimal assumption we have to make for local geodesics to exist: as in Riemannian geometry, those are $\mathcal{C}^1$ curves defined through a second order differential equation. Geodesics have constant speed, that is, $F(\dot{c}(t))$ is constant, and if $x = c(t)$ and $y = c(t')$ are points on the curve which are close enough, then the curve $c : [t, t'] \longrightarrow M$ is the shortest path from x to y:

$$d_F(x, y) = \int_t^{t'} F(\dot{c}(s))\ ds.$$

We say that the Finsler metric is complete if geodesics exist for all times. This is always the case if the manifold is compact. If the Finsler metric is complete, we can define the geodesic flow of the metric as the flow of the second-order differential equation which defines geodesics. In this way, the geodesic flow is defined on the tangent bundle TM. But, since geodesics have constant speed, this flow preserves the subbundles

$$T^{\alpha}M = \{u \in TM,\ F(u) = \alpha\},\ \alpha > 0.$$

The most natural space to study the geodesic flow is thus the unit tangent bundle T^1M. However, this space depends on the metric, so we will prefer considering this flow as defined on the homogeneous tangent bundle $HM = TM \smallsetminus \{0\}/\mathbb{R}_+^*$, identifying it with the unit tangent bundle: a point of the homogeneous tangent bundle consists of a base point on the manifold and a tangent direction at this point, that we denote by $w = (x, [\xi])$; its image $\varphi^t(w)$ by the geodesic flow is obtained by following during the time t the unit speed geodesic leaving x in the direction $[\xi]$.

Geodesic flows are important examples in dynamics, especially when the manifold is not "too big", so we can expect strong recurrence properties. For example, the geodesic flow of a compact negatively-curved Riemannian manifold has strong hyperbolic properties, such as the Anosov property. The main goal of this chapter is to study this kind of properties for the geodesic flow of compact quotients of some Finsler geometries, with an emphasis on Hilbert geometries.

2.3 Hilbert geometries and their isometries

A Hilbert geometry (Ω, d_Ω) is an example of a Finsler manifold. The Finsler norm of $u \in T_x\Omega$ is given by the formula

$$F(x,u) = \frac{|u|}{2}\left(\frac{1}{|xu^+|} + \frac{1}{|xu^-|}\right),$$

where $u^{\pm}$ are the intersection points of the line generated by u. It is Riemannian if and only if Ω is an ellipsoid [49], in which case (Ω, d_Ω) is the Beltrami model of the hyperbolic space.

Various volume forms can be associated to a Finsler metric. However, all natural volumes that could be associated to Hilbert geometries are equivalent (see L. Marquis' contribution to this volume). So let us fix the volume Vol_Ω once and for all as being the Busemann-Hausdorff volume: this is the Hausdorff measure of (Ω, d_Ω); equivalently, it is the volume such that the unit

Figure 2. The Finsler metric

Finsler ball has the same volume as the Euclidean unit ball.

Our main interest will lie in those Hilbert geometries which have enough symmetries to admit compact quotients. The rest of this part recalls some facts about the existence of such quotients.

2.3.1 Regular Hilbert geometries From the formula, we see that F is regular if the boundary $\partial\Omega$ of Ω is $\mathcal{C}^2$ with positive definite Hessian[1]. Geodesics coincide with projective lines. Thus, it is easy to see the geodesic flow on $H\Omega$: we just have to follow lines...
However, apart from the case of the ellipsoid, a regular Hilbert geometry has essentially no quotients:

Theorem 2.2 (É. Socié-Méthou [50]). *The isometry group of a regular Hilbert geometry (Ω, d_Ω) is compact, unless Ω is an ellipsoid.*

Hence, there is no interesting space where to study their geodesic flows from a global point of view. That is why we have to consider less regular Hilbert geometries.

2.3.2 Geodesics For a general Hilbert geometry, geodesics can be defined metrically: a geodesic segment is a metric isometry $c : [0, T] \longrightarrow \Omega$ from $\mathbb{R}$ to

[1] The Hessian is computed in some affine chart eqquiped with a Euclidean metric; its positive definiteness does not depend on the choice of the chart and the metric. See Section 8.5 for more on this.

(Ω, d_Ω) and a geodesic is an isometry from $\mathbb{R}$ to Ω. It is not difficult to see that projective lines are still geodesics but there might be others, even locally [22]. This happens as soon as the boundary of the convex set contains two nonempty open segments which are in a same 2-dimensional subspace but not in the same supporting hyperplane.

2.3.3 Isometries The *automorphism group* $\mathrm{Aut}(\Omega)$ consisting of those projective transformations preserving Ω is an important subgroup of isometries. Indeed, we expect $\mathrm{Aut}(\Omega)$ to be the full isometry group in most cases:

Conjecture. $\mathrm{Aut}(\Omega) = \mathrm{Isom}(\Omega, d_\Omega)$ *unless* (Ω, d_Ω) *is symmetric, in which case* $\mathrm{Aut}(\Omega)$ *is a subgroup of index* 2 *of* $\mathrm{Isom}(\Omega, d_\Omega)$.

Homogeneous Hilbert geometries are those whose automorphism group acts transitively. They have been described by M. Koecher and E. B. Vinberg in the fifties and sixties [37, 51].
Among them, *symmetric* ones are those which are self-dual. They fall into three classes: the simplices, the hyperbolic space and the symmetric spaces of the groups $\mathrm{SL}(n, K)$, with $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and $n \geqslant 3$ or the exceptional group $E_{6(-26)}$ (see [24] for example).

The last conjecture is confirmed in some cases:

- for strictly convex sets: it is a consequence of the uniqueness of geodesics (P. De La Harpe [22]);
- for polytopes: $\mathrm{Aut}(\Omega) = \mathrm{Isom}(\Omega, d_\Omega)$ unless Ω is a simplex, in which case $\mathrm{Aut}(\Omega)$ has index 2 in $\mathrm{Isom}(\Omega, d_\Omega)$ (P. De La Harpe [22] for the 2-dimensional case, B. Lemmens and C. Walsh [40] for the general case)
- for symmetric Hilbert geometries: A. Bosché [14] proved that $\mathrm{Aut}(\Omega)$ has index 2 in $\mathrm{Isom}(\Omega, d_\Omega)$.

A careful and general study of $\mathrm{Aut}(\Omega)$ is made in L. Marquis' contribution. In particular, he describes and classifies automorphisms in terms of the dynamics of their action on Ω.

2.3.4 Divisible Hilbert geometries

Definition 2.3. A quotient $M = \Omega/_{\Gamma}$ of a Hilbert geometry (Ω, d_Ω) by a discrete subgroup Γ of $\mathrm{Aut}(\Omega)$ is called a convex projective manifold (or orbifold in case Γ has torsion).
A Hilbert geometry (Ω, d_Ω) or the convex set Ω is said to be *divisible* if it admits a compact quotient by a discrete subgroup of $\mathrm{Aut}(\Omega)$.

Remark that by Selberg's lemma, if $\Gamma < \mathrm{Aut}(\Omega)$ has finite type, in particular when Γ acts cocompactly on Ω, then Γ has a finite index subgroup without torsion. In this case, the quotient $M = \Omega/_{\Gamma}$ is a smooth manifold and the group Γ is isomorphic to the fundamental group $\pi_1(M)$ of M via a faithful linear representation $\pi_1(M) \longrightarrow \mathrm{SL}(n+1,\mathbb{R})$.

Socié-Méthou's theorem asserts that the hyperbolic space is the only regular divisible Hilbert geometry. Among homogeneous Hilbert geometries, only the symmetric ones and their products are divisible [52]. Remark that among symmetric Hilbert geometries, the hyperbolic space is the only one to be strictly convex or with $\mathcal{C}^1$ boundary.

Apart from the homogeneous cases, the existence of divisible Hilbert geometries has been a long-standing question. From a general point of view, the following dichotomy result of Y. Benoist is essential:

Theorem 2.4 (Y. Benoist [10]). *Let (Ω, d_Ω) be a divisible Hilbert geometry, divided by a discrete subgroup $\Gamma < \mathrm{Aut}(\Omega)$. The following propositions are equivalent:*

- *the convex set Ω is strictly convex;*
- *the boundary $\partial\Omega$ is $\mathcal{C}^1$;*
- *the metric space (Ω, d_Ω) is Gromov-hyperbolic;*
- *the group Γ is Gromov-hyperbolic.*

Recall that a geodesic metric space (X, d) is said to be *Gromov-hyperbolic* if there is some $\delta > 0$ such that any geodesic triangle $xyz \subset X$ of vertices $x, y, z \in X$ is δ-thin, that is, for any point p on the side $[xz]$,

$$\min\{d(p,[xy]), d(x,[yz])\} \leqslant \delta.$$

When the metric space is Gromov-hyperbolic for the constant δ, we say it is δ-hyperbolic.
A finitely-generated group Γ is Gromov-hyperbolic if its Cayley graph with respect to some finite set of generators is Gromov-hyperbolic for the word metric. This property does not depend on the generating set, but the constant δ of hyperbolicity does depend on it.

2.3.5 Strictly convex divisible Hilbert geometries Examples of strictly convex divisible Hilbert geometries are now available in all dimensions. They are obtained by deformations of compact hyperbolic manifolds following an idea of [32].
In low dimensions, examples can be obtained using Coxeter groups. First examples were provided by these means [33]. Y. Benoist [9] also constructed

in this way strictly convex divisible Hilbert geometries which are not quasi-isometric to the hyperbolic space. M. Kapovich [34] found such examples in all dimensions.
We refer to L. Marquis' contribution for a deeper study of this question.

Assume $M = \Omega/_{\Gamma}$ is a compact quotient manifold of a strictly convex Hilbert geometry. Then all elements $g \in \Gamma$ are *hyperbolic isometries* of (Ω, d_Ω), which means the following:

- g fixes exactly two points x_g^+ and x_g^- on the boundary $\partial\Omega$ and acts as a translation (for the Hilbert metric) on the line $(x_g^- x_g^+)$, which is called *the axis of* g;
- the point x_g^+ is attractive: for any $y \in \overline{\Omega} \smallsetminus \{x_g^-\}$, we have $\lim\limits_{n\to+\infty} g^n y = x_g^+$;
- the point x_g^+ is repulsive: for any $y \in \overline{\Omega} \smallsetminus \{x_g^+\}$, we have $\lim\limits_{n\to+\infty} g^{-n} y = x_g^-$.

As elements of $\mathrm{SL}(n+1,\mathbb{R})$, the elements $g \in \Gamma$ are biproximal: their biggest and smallest eigenvalues (in modulus) λ_0 and λ_n are simple, that is, the corresponding eigenspaces are 1-dimensional; these eigenspaces are the fixed points x_g^+ and x_g^-; the translation distance on the axis is $\frac{1}{2}\log\frac{\lambda_0}{\lambda_n}$.

Let us end this paragraph with a result about the group Γ which divides the Hilbert geometry, that will be crucial to deduce dynamical rigidity results. It reads as

Theorem 2.5 (Y. Benoist [6]). *Let (Ω, d_Ω) be a strictly convex Hilbert geometry, divided by a discrete subgroup $\Gamma < \mathrm{Aut}(\Omega)$. The group Γ is Zariski-dense in $\mathrm{SL}(n+1,\mathbb{R})$, unless Ω is an ellipsoid.*

Recall that the *Zariski-closure* of a subgroup Γ of $\mathrm{SL}(n+1,\mathbb{R})$ is the smallest algebraic subgroup G of $\mathrm{SL}(n+1,\mathbb{R})$ which contains Γ. We then say that Γ is Zariski-dense in G.
The hypothesis of strict convexity in the last theorem is actually unnecessary, but the proof in this case is far more involved [8].
This last theorem will be useful through the following characterization of Zariski-dense subgroups of semisimple Lie groups, which is also due to Y. Benoist. Let us explain it in the context of the Lie group $\mathrm{SL}(n+1,\mathbb{R})$.
To each element g in $\mathrm{SL}(n+1,\mathbb{R})$, we associate the vector

$$\ln(g) = (\ln\lambda_0(g), \cdots, \ln\lambda_n(g)) \in \mathbb{R}^{n+1},$$

where $\lambda_0(g) \geqslant \lambda_1(g) \geqslant \cdots \geqslant \lambda_n(g)$ denote the moduli of the eigenvalues of g.
For a subgroup Γ of $\mathrm{SL}(n+1,\mathbb{R})$, let $\ln\Gamma = \{\ln g,\ g \in \Gamma\}$.

Theorem 2.6 (Y. Benoist, [7]). *Let* Γ *be a subgroup of* $\mathrm{SL}(n+1,\mathbb{R})$*. If* Γ *is Zariski-dense in* $\mathrm{SL}(n+1,\mathbb{R})$*, then the subgroup generated by* $\ln\Gamma$ *is dense in the subspace* $\{\sum_i x_i = 0\}$.

2.4 The geodesic flow

2.4.1 Definition For general Hilbert geometries, we consider the geodesic flow $\varphi^t : H\Omega \longrightarrow H\Omega$ following projective lines. Let $X : H\Omega \longrightarrow TH\Omega$ be the vector field which generates this flow. If we choose an affine chart and a Euclidean metric $|\cdot|$ on it, in which Ω is a bounded open convex set, then X can be written as $X = mX^e$, where X^e is the generator of the Euclidean geodesic flow, because X and X^e have the same orbits. A direct computation gives

$$m(w) = 2\left(\frac{1}{|xw^+|} + \frac{1}{|xw^-|}\right)^{-1} = 2\,\frac{|xw^+|\,|xw^-|}{|w^+w^-|},\ w = (x,[\xi]) \in H\Omega,$$

where w^+ (resp. w^-) denotes the intersection point of $\partial\Omega$ with the ray leaving x in the direction ξ (resp. $-\xi$). This link between X and X^e, that is, the flatness of Hilbert geometries, is crucial in extending some differential objects in Section 3.
The vector field X and the geodesic flow φ^t have the same regularity as the boundary of Ω.

If $M = \Omega/\Gamma$ is a quotient manifold, the geodesic flow is defined on HM by projection. To study its (local) properties, we will often work directly on $H\Omega$ where geodesics are lines and computations can be made.
For a compact quotient, we can expect from Y. Benoist's theorem strong differences for the geodesic flow on HM between the strictly convex (with $\mathcal{C}^1$ boundary) and non-strictly convex (with non-$\mathcal{C}^1$ boundary) cases. We can give easy illustrations of this difference:

- at a non-$\mathcal{C}^1$ point of the boundary, there are asymptotic geodesics whose distance does not go to 0;
- if I is an open segment in the boundary $\partial\Omega$, the distance between two geodesic rays ending at some points of I stays bounded.

We do not know much more than these observations in the non-strictly convex case. On the contrary, more attention has been paid to the strictly convex case, where the flow happens to exhibit strong hyperbolicity properties. This chapter is dedicated to this case, that is, the study of the geodesic flow of a compact quotient $M = \Omega/\Gamma$ with Ω strictly convex (and by Theorem 2.4 its boundary is of class $\mathcal{C}^1$).

2.4.2 Closed orbits Assume $M = \Omega/_{\Gamma}$ is any quotient manifold of a strictly convex (with $\mathcal{C}^1$ boundary) Hilbert geometry (Ω, d_Ω), with Γ a discrete subgroup of $\mathrm{Aut}(\Omega)$. We use the same notation on Ω and M, on $H\Omega$ and HM.

The *orbit* of a point $w \in H\Omega$ is the set $\varphi \cdot w = \{\varphi^t(w),\ t \in \mathbb{R}\}$. The orbit $\varphi \cdot w$ of $w \in HM$ is *closed* if there exists $T > 0$ such that $\varphi^T(w) = w$. The smallest $T > 0$ which satisfies this equality is the *length of the orbit.* We also say that w is a *periodic point* and that T is its *period.*
If G is a group, a non-identity element g of G will be called *primitive* if there is no $h \in G$ and $k \geqslant 2$ such that $g = h^k$. The same will be said of the conjugacy class of g in G. For example, if $[g]$ is a free homotopy class in M, that is, the conjugacy class of an element g in the group $\pi_1(M)$, $[g]$ is primitive if "it does not make more than one loop".

A closed orbit on HM lifts to an orbit on $H\Omega$ which projects down in Ω onto an *oriented* line (x^-x^+), $x^-, x^+ \in \partial\Omega$, which is left invariant by some non-identity hyperbolic elements of Γ. Among such elements, which are all hyperbolic of axis (x^-x^+), only one is primitive and has x^+ as attractive fixed point and x^- as repulsive fixed point. In this way, we can associate to each closed orbit a primitive hyperbolic element of Γ. Conversely, to such an element is associated the oriented line (x^-x^+), which yields a closed orbit of the geodesic flow. Two such elements will define the same closed orbit if they are conjugate. Therefore we have the

Proposition 2.7 (see [10], Proposition 5.1). *Let (Ω, d_Ω) be a strictly convex Hilbert geometry with $\mathcal{C}^1$ boundary and $M = \Omega/_{\Gamma}$ a compact quotient manifold, where Γ a discrete subgroup of* $\mathrm{Aut}(\Omega)$*.*
Closed orbits of the geodesic flow on HM are in bijection with conjugacy classes of primitive hyperbolic elements of Γ*.*

A closed orbit on HM gives by projection on M a closed geodesic, provided with an orientation. If g is a primitive hyperbolic element of Γ, the orbits defined by g and its inverse g^{-1} project down on the same closed geodesic in M, the closed orbits differ only by the direction in which they run along the geodesic.
A direct computation shows that the length of the associated orbit is $\frac{1}{2}\log\frac{\lambda_0}{\lambda_n}$, where λ_0 and λ_n are the moduli of the biggest and smallest eigenvalues of g.

3 Differential objects in Finsler and Hilbert geometries

Given a Riemannian manifold, its geodesic flow benefits a lot from the rich geometric structure of the manifold, and this can be extended to regular Finsler metrics. For less regular Hilbert geometries, this is much more delicate; only part of the objects can be constructed: these defects can be seen as an explanation for some rigidity results we will present later on.

3.1 The Hilbert 1-form

For a regular Finsler metric, the geodesic flow is the Reeb flow of a 1-form A called the Hilbert 1-form. This is Proposition 3.2 below. The Hilbert 1-form is defined via the vertical derivative of the Finsler metric:

Definition 3.1. Let F be a $\mathcal{C}^1$ Finsler metric on a manifold M. Denote by $p : TM \longrightarrow M$ and $r : TM \smallsetminus \{0\} \longrightarrow HM$ the canonical bundle projections. The *vertical derivative* of F is the 1-form d_vF on TM defined by

$$d_vF(x,\xi)(Z) = \lim_{\varepsilon\to 0} \frac{F(x,\xi+\varepsilon dp(Z)) - F(x,\xi)}{\varepsilon}, \ (x,\xi)\in TM, \ Z \in T_{(x,\xi)}TM. \tag{3.1}$$

It descends by homogeneity on HM to give the *Hilbert* 1*-form* A of F, that is,

$$A = r * d_vF.$$

The definition of A is made possible because the 1-form d_vF depends only on the direction $[\xi]$: it is invariant under the flow generated by the vector field $D = \sum \xi_i \frac{\partial}{\partial \xi_i}$. Let $\pi : HM \longrightarrow M$ denote the bundle projection. Since $d\pi(X(x,[\xi])) \in [\xi]$ and $F(d\pi(X(x,[\xi]))) = 1$, we can write

$$A(Z) = \lim_{\varepsilon\to 0} \frac{F(x, d\pi(X-\varepsilon Z)) - 1}{\varepsilon}.$$

Proposition 3.2. *Let F be a regular Finsler metric on a manifold M and A its Hilbert* 1*-form. The generator X of the geodesic flow of F is the unique solution of*

$$A(X) = 1; \ dA(X,\cdot) = 0.$$

Moreover, the geodesic flow preserves the volume form $A\wedge dA^{n-1}$, that we call the Liouville volume or Liouville measure.

There is another way of seeing A as we we explain in the next section. We learned this from T. Barthelmé, and more on this question can be found in his PhD thesis [5]; Arnold's book [3] is still a good reference for the classical facts on Hamiltonian dynamics.

3.2 The Legendre transform

3.2.1 Definition for regular metrics We can see the geodesic flow of a regular Finsler metric as a Hamiltonian flow by using the *Legendre transform*.

Definition 3.3. Let F be a regular Finsler metric on a manifold M. The Legendre transform $\mathcal{L}_F : TM \longrightarrow T^*M$ is defined by the formula

$$\mathcal{L}_F(v)(u) = \frac{1}{2}\frac{d}{dt}|_{t=0}F^2(x, v+tu). \tag{3.2}$$

For a Riemannian metric, the Legendre transform is linear: for a vector $v \in T_xM$, the Legendre transform of v is the dual 1-form defined by v:

$$\mathcal{L}_F(v)(u) = \langle v, u\rangle,\ u \in T_xM.$$

For a regular Finsler metric, we can see the Legendre transform geometrically in the following way. Let $B_x(r)$ and $S_x(r)$ be the metric ball and sphere of radius $r > 0$ of the Finsler norm $F(x, .)$ on T_xM. The Legendre transform $\mathcal{L}_F(v)$ of a vector $v \in S_x(r)$ is then the 1-form such that

$$\mathcal{L}_F(v)(v) = F(v)^2;\ \ker \mathcal{L}_F(v) = T_vS_x(r).$$

3.2.2 Finsler cometrics

Definition 3.4. A *Finsler cometric* is a function $F^* : T^*M \longrightarrow [0, +\infty)$ such that:

- $F^*(x, \lambda\alpha) = \lambda F^*(x, \alpha)$, $(x, \alpha) \in T^*M$, $\lambda \geqslant 0$;
- $F^*(x, \alpha+\beta) \leqslant F^*(x, \alpha) + F^*(x, \beta)$, $x \in M$, $\alpha, \beta \in T^*_xM$.

We will say that a Finsler cometric F^* is *regular* if F^* is $\mathcal{C}^2$ and the boundary of its unit coballs have positive definite Hessian.
Its Legendre transform $\mathcal{L}_{F^*} : T^*M \longrightarrow TM$ is defined by

$$\mathcal{L}_{F^*}(v)(u) = \frac{1}{2}\frac{d}{dt}|_{t=0}(F^*)^2(x, v+tu). \tag{3.3}$$

To a Finsler metric F is naturally associated a Finsler cometric F^*; this is the usual dual norm, defined as

$$F^*(x, \alpha) = \max\{\alpha(v),\ v \in S_x(1)\}.$$

If F is regular, the Finsler cometric F^* is also regular.

The Legendre transform is then a $\mathcal{C}^1$-diffeomorphism from TM to T^*M which sends F to F^*: for all $v \in v$, $F^*(\mathcal{L}_F(v)) = F(v)$. Its inverse is the Legendre transform $\mathcal{L}_{F^*}$ of F^*. $\mathcal{L}_F$ is trivially homogeneous of degree 1, that is $\mathcal{L}_F(\lambda v) = \lambda\mathcal{L}_F(v)$, so it defines a map l_F from HM to $H^*M = T^*M \smallsetminus \{0\}/\mathbb{R}^*_+$, that we still call the Legendre transform. Its inverse is the map l_{F^*}.

3.2.3 The Hilbert 1-form from the Hamiltonian point of view The cotangent space T^*M is canonically a symplectic manifold, the symplectic form ω being given by the exterior derivative of the Liouville form $L = \sum \xi_i dx_i$. The geodesic flow on T^*M is the Hamiltonian flow of the Hamiltonian function $H(x,\alpha) = \frac{1}{2}(F^*(x,\alpha))^2$: this is the flow generated by the vector field X_H such that

$$dH(Z) = \omega(X_H, Z),\ \ Z \in TTM.$$

Let $S^*M = \{u \in T^*M,\ F^*(u) = 1\}$ denote the unit sphere bundle of F^*. The restriction of the projection $T^*M \longrightarrow H^*M$ to S^*M, is a diffeomorphism that we denote by $p_{F^*} : S^*M \longrightarrow H^*M$. Through this map, we can define on H^*M the 1-form $p_{F^*}L$, the 2-form $p_{F^*}\omega$ and the vector field $p_{F^*}X_H$, which are the push-forward under p_{F^*} of the restrictions of L, ω and X_H to S^*M. The 2-form $p_{F^*}\omega$ is invariant by the flow of $p_{F^*}X_H$, and the Liouville measure of this flow is the invariant volume given by $p_{F^*}L \wedge (p_{F^*}\omega)^{n-1}$.

Proposition 3.5. *Let F be a regular Finsler metric on a manifold M.*
The Hilbert form A, the 2-form dA and the Liouville volume $A \wedge dA^{n-1}$ are the respective pullbacks by l_F of the Liouville form $p_{F^*}L$, the symplectic form $p_{F^*}\omega$, and the volume $p_{F^*}L \wedge (p_{F^*}\omega)^{n-1}$.
The Legendre transform l_F also conjugates the flow of $p_{F^*}X_H$ on H^*M and the geodesic flow of F on HM.

3.2.4 General Finsler metrics For a general Finsler metric, the previous constructions have in general no meaning. Only the Finsler cometric F^* is well defined. Using the geometrical construction, the Legendre transform could also be seen as a multiple-valued function: for example, to a unit vector $v \in T_xM$ where the unit sphere $S_x(1)$ is not $\mathcal{C}^1$, the Legendre transform would associate the set of linear forms α such that $\alpha(v) = 1$ and whose kernel is one of the supporting hyperplanes of $S_x(1)$ at v.

If the Finsler metric is $\mathcal{C}^1$ with strictly convex unit balls, the unit balls of the Finsler cometric F^* are also strictly convex with $\mathcal{C}^1$ boundary. The Legendre transforms $\mathcal{L}_F$ and l_F are then well defined homeomorphisms through the geometrical construction, but they are no more of class $\mathcal{C}^1$. Thus, even if the differential forms L, ω and the vector field X_H are well defined on T^*M, as well as their projections by (the now $\mathcal{C}^1$-map) p_{F^*} on H^*M, we cannot pull these objects back to HM by the Legendre transform. However, I do not know if we can define a geodesic flow in this context.

3.3 Connection and parallel transport

Given a manifold M, there are many ways to identify two distinct tangent spaces T_xM and T_yM. If M is the affine space $\mathbb{R}^n$, then the most natural way is to identify the tangent spaces to the space itself. For a general manifold, such an identification is called a *linear connection*: any path c from x to y gives an identification, that is a linear isomorphism, between T_xM and T_yM via *parallel transport*.

When M is a Riemannian manifold, we would like the linear isomorphisms to be actual isometries between the tangent spaces. In other words, we want the Riemannian metric to be parallel. The Riemannian or Levi-Civita connection is the unique linear connection without torsion for which the metric is parallel.

This construction cannot extend to the regular Finsler context. As we already said, the good geometrical space for a Finsler metric is the tangent bundle TM and not the manifold M. A "Finsler" connection should then be an object defined on TM, which would extend the Levi-Civita connection, in the sense that this one should be recovered "by projection" in the case of a Riemannian metric. However, depending on which properties we want the connection to have (such as no torsion, parallelism of the Sasaki metric), we will obtain different connections.
A way of avoiding these problems was discovered by P. Foulon: if we restrict ourselves to transport along geodesics then we get a well defined linear transport. The good space to look at is then HM, and the good object is the geodesic flow. In this context, we can define in an intrinsic and canonical way linear objects generalizing the Riemannian ones.

Foulon's contribution appeared in [25], but one can find a shortest version in English in [27]. Here, we just describe the conclusions, which apply in general for $\mathcal{C}^3$ regular Finsler metrics.
The vertical bundle of HM is the bundle $VHM = \ker d\pi$, where π is the canonical bundle projection $\pi : HM \longrightarrow M$. The tangent bundle to HM admits then a horizontal-vertical decomposition as

$$THM = \mathbb{R} \cdot X \oplus h^F HM \oplus VHM. \tag{3.4}$$

(The vector field X is the generator of the geodesic flow.) Furthermore, there is a pseudo-complex structure on $h^F HM \oplus VHM$ which exchanges vertical and horizontal subspaces; that is, a bundle isomorphism

$$J^F : h^F HM \oplus VHM \longrightarrow h^F HM \oplus VHM$$

such that

$$J^F(VHM) = h^F HM, J^F(h^F HM) = VHM \text{ and } J^F \circ J^F = -Id.$$

Associated to this decomposition is a "partial" covariant derivative D^X which defines the differential of a vector field Z defined along an orbit of the geodesic flow in the direction of the flow. This is a differential operator of order 1 $D^X : THM \longrightarrow THM$ which commutes with J^F, and satisfies $D^X(X) = 0$. A vector field Z defined along an orbit of the flow is parallel if $D^X(Z) = 0$.
This allows us to define the *parallel transport* of a vector $Z(w) \in T_w HM$ along the orbit $\varphi \cdot w$ of w: it is the unique parallel vector field Z defined on $\varphi \cdot w$ whose value at w is $Z(w)$. If we fix a $t \in \mathbb{R}$, we will denote by $T^t(Z(w)) := Z(\varphi^t(w))$ the parallel transport of $Z(w)$ during the time t. This yields a bundle isomorphism $T^t : THM \longrightarrow THM$ which sends $T_w HM$ on $T_{\varphi^t(w)} HM$.
Since D^X commutes with J^F, the parallel transport T^t also commutes with J^F and, since and X is parallel, it preserves the decomposition $THM = \mathbb{R} \cdot X \oplus h^F HM \oplus VHM$.
The projection $d\pi : THM \longrightarrow TM$ induces an isomorphism between the space $\mathbb{R} \cdot X(w) \oplus h^F HM(w)$ and $T_{\pi(w)} M$ for each $w \in HM$, so we can define a parallel transport along geodesics on M. Let $c : \mathbb{R} \longrightarrow M$ be a geodesic and $x = c(0)$, $w = [\dot{c}(0)] \in HM$. If $u \in T_x M$, we consider its lift $U(w) = d\pi^{-1}(u)$ to $\mathbb{R} \cdot X(w) \oplus h^F HM(w)$, and define the parallel transport of u along the geodesic c by

$$T_c^t(u) = d\pi(T^t(U(w))),\ t \in \mathbb{R}.$$

This gives a linear isomorphism T^t between the tangent spaces $T_x M$ and $T_{c(t)} M$.

3.4 Curvature and Jacobi fields

The *Jacobi operator* R^F is defined by

$$R^F(X) = 0,\ R^F(Y) = p_v^F([X, J^F(Y)]),\ R^F \text{ commutes with } J^F,$$

where $p_v^F(Z)$ denotes the projection on the vertical subspace VHM of the vector $Z \in THM$, with respect to the horizontal-vertical decomposition 3.4.
In the case X is the geodesic flow of a Riemannian metric g on M, the Jacobi operator allows to recover the curvature tensor R^g of g: for $u, v \in T_x M \setminus \{0\}$, we have

$$R^g(u,v)u = \frac{d\pi(R^F V(x,[u]))}{g(u,u)},$$

where $V(x,[u])$ is the unique vector in $\mathbb{R}\cdot X(x,[u]) \oplus h^F HM(x,[u])$ such that $d\pi(V(x,[u])) = v$.

Definition 3.6. We will say that a regular Finsler manifold (M,F) is *negatively curved* if the Jacobi operator R^F is negative definite.

A *Jacobi field* is a vector field J on HM which satisfies

$$D^X D^X J + R^F J = 0.$$

Jacobi fields form a $2(2n-1)$-dimensional vector space, which is J^F-invariant. In the Riemannian case, one recovers the usual Jacobi fields by projection on the base. Jacobi fields on HM do not contain more information than the ones on M, the projection only rubs out the J^F-symmetry between horizontal and vertical subspaces, as for the tensor curvature.

3.5 Metrics on HM

For a Riemannian metric g on M, there is a canonical associated Riemannian metric on HM called the Sasaki metric. To define it, we use the Levi-Civita connection $THM = \mathbb{R}\cdot X \oplus VHM \oplus h^g HM$ and do the following construction.

- The decomposition $THM = \mathbb{R}\cdot X \oplus VHM \oplus h^g HM$ is orthogonal for the Sasaki metric.
- Each subspace $\mathbb{R}\cdot X(u) \oplus h^g_u HM$ is isomorphic to the tangent space $T_{\pi(u)}M$ of M via the projection π; the quadratic form on $\mathbb{R}\cdot X(u) \oplus h^g HM(u)$ is defined as the pullback of $g_{\pi(u)}$ by π.
- Vertical and horizontal subspaces are identified by the complex structure J^g; the quadratic form on $V_u HM$ is the push-forward by J^g of the quadratic form on $h^g_u TM$.

By construction, the Sasaki metric is invariant by the pseudo-complex structure, and vertical and horizontal subspaces are isometric.

There are different ways of extending this definition to the Finsler context.

Usually, Finsler geometers consider the following generalization for regular Finsler metrics. The Sasaki metric on HM associated to a regular Finsler metric F on M is the Riemannian metric g^F defined by

$$g^F_{(x,[u])}(Y,Y') = \mathrm{Hess}_{(x,u)}(F^2)(Y,Y'),\ Y,Y' \in VHM,$$

where u is the unit vector in $[u]$, and

$$g^F(X,X) = 1,\ g^F(h,h') = g^F(J^F(h),J^F(h')),\ h,h' \in h^F HM.$$

Equivalently, the formula on VHM can be replaced by

$$g^F(Y, Y') = dA([X, Y], Y'),\ Y, Y' \in VHM,$$

where A is the Hilbert 1-form introduced above.

Another way would be the following, but this would yield a Finsler metric on HM: we let, for $Z = aX + Y + h$,

$$\|Z\| = \left(|a|^2 + (F(d\pi h))^2 + (F(d\pi J^F Y))^2\right)^{1/2}.$$

When the metric is Riemannian, we recover the Sasaki metric. When the metric is Finsler, this metric is only a Finsler metric (except for surfaces), whose regularity is one less than the original one. The complex structure is still an isometry between vertical and horizontal subspaces.
In fact, I do not think the metric $\|\cdot\|$ was already considered before. However, the first construction does not make sense for non-regular Finsler metrics and that is the reason why we consider the second one. In particular, we will use this second metric to study non-regular Hilbert geometries.

3.6 The case of Hilbert geometries

The above constructions work in general for at least $\mathcal{C}^3$ regular Finsler metrics. It is however possible to exploit the flatness of Hilbert geometries to extend this formalism to the case where the convex set is strictly convex with $\mathcal{C}^1$ boundary. Choose an affine chart adapted to the properly convex set $\Omega \subset \mathbb{RP}^n$ and fix a Euclidean metric on it. Denote by $X^e : H\Omega \longrightarrow H\Omega$ the generator of the Euclidean geodesic flow. The crucial thing is the link $X = mX^e$ which exists between the geodesic flow of the Hilbert and the Euclidean metrics and the fact that m and its first derivatives are smooth in the direction of the flow(s). We can find details in [20].
It allows us to use freely Foulon's dynamical formalism, that is the objects defined in sections 3.3 and 3.4. For example, we can prove the following proposition, which is immediate for regular metrics by using the differential dA of the Hilbert form:

Proposition 3.7 ([20]). *Let Ω be a strictly convex subset of $\mathbb{RP}^n$ with $\mathcal{C}^1$ boundary and A the Hilbert form of the Hilbert metric F on Ω. Then*

(i) $\ker A = VH\Omega \oplus h^F H\Omega$;

(ii) *A is invariant under the geodesic flow.*

A general result linking R^X and R^{X^e} and a direct computation prove that such Hilbert geometries have constant curvature -1:

Proposition 3.8. *Let Ω be a strictly convex subset of $\mathbb{RP}^n$ with $\mathcal{C}^1$ boundary. The Jacobi endomorphism of the Hilbert geometry (Ω, d_Ω) is*

$$R^F = -Id$$

on $h^F H\Omega + VH\Omega$.

However, the interpretation of such a result is not the same as in Riemannian geometry. We should not forget that the good geometrical space in Finsler geometry is the tangent bundle and not the manifold.
Perhaps, the main consequence of this fact (at least in the context of this chapter) is an explicit computation of Jacobi fields:

Lemma 3.9. *Let (Ω, d_Ω) be a Hilbert geometry defined by a strictly convex set with $\mathcal{C}^1$ boundary. Let $w \in H\Omega$, and $J_w \in VH\Omega + h^F H\Omega$. Then the Jacobi field J along the orbit of w such that $J(w) = J_w$ and $D^X J(w) = \pm J_w$ is given by*

$$J(\varphi^t w) = e^{\pm t} T^t J_w.$$

This describes the behaviour of all Jacobi fields since for any $Z_w, Z'_w \in TH\Omega$, we can write

$$Z_w = \lambda X(w) + Z_w^+ + Z_w^-,\ Z'_w = \lambda' X(w) + Z_w^+ - Z_w^-,$$

with

$$Z_w^+ = \frac{Z_w + Z'_w}{2},\ Z_w^- = \frac{Z_w - Z'_w}{2}.$$

Hence the Jacobi field J along the orbit of w such that $J(w) = Z_w$ and $D^X J(w) = Z'_w$ will be given by

$$J(\varphi^t w) = (\lambda' t + \lambda) X(\varphi^t w) + e^t T^t Z_w^+ + e^{-t} T^t Z_w^-.$$

4 Stable and unstable bundles and manifolds

We assume in this section that the convex set Ω which defines the Hilbert geometry is strictly convex with $\mathcal{C}^1$ boundary. The geodesic flow $\varphi^t : H\Omega \longrightarrow H\Omega$ is then a $\mathcal{C}^1$ flow. We want to study the (spatially infinitesimal) behaviour of the geodesic flow on the manifold $H\Omega$ equipped with the Finsler metric $\|\cdot\|$ introduced in Section 3.5. We denote by $d_{H\Omega}$ the metric induced by $\|\cdot\|$ on $H\Omega$.
Even if we will only consider compact quotients in the other sections, we do not make here any assumption on the existence of a quotient manifold, aiming to provide objects and results that could be useful for more general quotients.

4.1 Busemann functions and horospheres

The *Busemann function* based at $\xi \in \partial\Omega$ is defined by

$$b_\xi(x,y) = \lim_{p\to\xi} d_\Omega(x,p) - d_\Omega(y,p),$$

which, in some sense, measures the (signed) distance from x to y in Ω as seen from the point $\xi \in \partial\Omega$. A particular expression for b is given by

$$b_\xi(x,y) = \lim_{t\to+\infty} d_\Omega(x,\gamma(t)) - t,$$

where γ is the geodesic leaving y at $t=0$ to ξ. When ξ is fixed, then b_ξ is a surjective map from $\Omega\times\Omega$ onto $\mathbb{R}$. When x and y are fixed, then $b_.(x,y) : \partial\Omega \to \mathbb{R}$ is bounded by a constant $C = C(x,y)$.

The *horosphere* passing through $x \in \Omega$ and based at $\xi \in \partial\Omega$ is the set

$$\mathcal{H}_\xi(x) = \{y \in \Omega,\ b_\xi(x,y) = 0\}.$$

$\mathcal{H}_\xi(x)$ is also the limit when p tends to ξ of the metric spheres $S(p, d_\Omega(p,x))$ about p passing through x. In some sense, the points on $\mathcal{H}_\xi(x)$ are those which are as far from ξ as x is.
The (open) horoball $H_\xi(x)$ defined by $x \in \Omega$ and based at $\xi \in \partial\Omega$ is the "interior" of the horosphere $\mathcal{H}_\xi(x)$, that is, the set

$$H_\xi(x) = \{y \in \Omega,\ b_\xi(x,y) > 0\}.$$

It is easy to see that horospheres have the same kind of regularity as the boundary of Ω.

A consequence of (the proof of) Proposition 3.7 is the following

Corollary 4.1. *Let $w = (x,[\xi]) \in H\Omega$, $w^\pm = \varphi^{\pm\infty}(w) \in \partial\Omega$ and ξ the unit vector in $[\xi]$. The projection $d\pi(VH\Omega + h^F H\Omega)$ is the tangent space at x to both $\mathcal{H}_{w^+}(x)$ and $\mathcal{H}_{w^-}(x)$ and the tangent space at ξ to the unit sphere of $F(x,.)$ in $T_x\Omega$.*

4.2 Stable and unstable bundles and manifolds

Recall the

Definition 4.2. The stable set of $w \in H\Omega$ is the set of points $v \in H\Omega$ such that $\lim_{t\to+\infty} d(\varphi^t w, \varphi^t v) = 0$. The unstable set is the set of points $v \in H\Omega$ such that $\lim_{t\to-\infty} d(\varphi^t w, \varphi^t v) = 0$.

Knowing a bit of hyperbolic geometry, it is not difficult to "find" what should be the stable and unstable sets of a point $w = (x, [\xi]) \in H\Omega$. Because $d \geqslant d_\Omega \circ \pi$, we see that the extreme point of the orbit of a point $v \in W^s(w)$ must coincide with the one for w, that is $v^+ = w^+$. Moreover, the $\mathcal{C}^1$ regularity of the boundary $\partial\Omega$ implies that the horosphere $\mathcal{H}_{w^+}(\pi(w))$ about w^+ through $\pi(w)$

$$\mathcal{H}_{w^+}(\pi(w)) = \pi\left(\{v \in H\Omega,\ d_\Omega(\pi(\varphi^t(w)), \pi(\varphi^t(v)) = 0\}\right).$$

Thus, the stable set of w has to be a subset of the set $W^-(w)$ defined as the set of points v such that $v^+ = w^+$ and whose projection $\pi(v)$ is on the horosphere through w about w^+:

$$W^-(w) = \{v \in H\Omega \mid v^+ = w^+,\ \pi(v) \in \mathcal{H}_{w^+}(\pi(w))\}.$$

Similarly, the unstable set of w has to be a subset of the set

$$W^+(w) = \{v \in H\Omega \mid v^- = w^-,\ \pi(v) \in \mathcal{H}_{w^-}(\pi(w))\}.$$

Figure 3. The submanifolds $W^+(w)$ and $W^-(w)$, $w = (x, [\xi])$

Both sets $W^-(w)$ and $W^+(w)$ are $\mathcal{C}^1$ submanifolds of $H\Omega$. The sets $W^-(w)$, $w \in H\Omega$, foliate $H\Omega$, as well as the sets $W^+(w)$; in general, these foliations are only $\mathcal{C}^0$. We will refer to them as the $-$ and $+$ foliations.
We immediately see that both foliations are invariant under the geodesic flow:

$$W^-(\varphi^t(w)) = \varphi^t(W^-(w))\text{and } W^+(\varphi^t(w)) = \varphi^t(W^+(w)).$$

The rest of this section is dedicated to prove the following

Theorem 4.3. *Let $w \in H\Omega$. The sets $W^-(w)$ and $W^+(w)$ are the stable and unstable sets of w. The unstable and stable tangent bundles are given by*

$$E^u = \{Y + J^F(Y), Y \in VH\Omega\}$$

and

$$E^s = \{Y - J^F(Y), Y \in VH\Omega\} = J^F(E^u).$$

4.2.1 A temporary Finsler metric Let E^- and E^+ be the tangent bundles to the $-$ and $+$ foliations. These bundles define a φ^t-invariant decomposition of the tangent bundle

$$THM = \mathbb{R} \cdot X \oplus E^- \oplus E^+.$$

Corollary 4.1 implies that $E^- \oplus E^+ = VH\Omega + h^F H\Omega$.

We define a temporary Finsler metric $\|\cdot\|^{\pm}$ by

$$\|Z\|^{\pm} = \left(|a|^2 + 4(F(d\pi Z^+)^2 + F(d\pi Z^-)^2)\right)^{1/2}.$$

It is not difficult to see that $W^+(w)$ and $W^-(w)$ are the stable and unstable sets of w for this metric: this is what Corollary 4.5 below asserts.

We first need to make a little computation. To simplify this computation, we will use the projective nature of our objects and choose a good affine chart and a good Euclidean metric on it.
Let $w = (x, [\xi]) \in H\Omega$. *A good chart at w* is an affine chart where the intersection $T_{w^+}\partial\Omega \cap T_{w^-}\partial\Omega$ is contained in the hyperplane at infinity, and a Euclidean structure on it so that the line (xw^+) is orthogonal to $T_{w^+}\partial\Omega$ and $T_{w^-}\partial\Omega$ (see Figure 4).

Lemma 4.4. *Let $w \in H\Omega$, $Z^- \in E^-(w)$ and fix a good chart at w. Set $x = \pi(w)$, $x_t = \pi\varphi^t(w)$, $z = \pi(Z^-)$, $z_t = d\pi d\varphi^t(Z^-)$. We have*

$$\|d\varphi^t(Z^-)\|^{\pm} = 2F(z_t) = \frac{|z|}{|xw^+|}\left(\frac{|x_t w^+|}{|x_t z_t^+|} + \frac{|x_t w^+|}{|x_t z_t^-|}\right).$$

Proof. We have $\|d\varphi^t(Z^-)\|^{\pm} = 2F(z_t)$ by definition of the metric $\|\cdot\|^{\pm}$. Now, by definition of F, we get

$$F(z_t) = \frac{|z_t|}{2}\left(\frac{1}{|x_t z_t^+|} + \frac{1}{|x_t z_t^-|}\right),$$

where z_t^+ and z_t^- are the intersection points of the line $\{x_t + \lambda z_t,\ \lambda \in \mathbb{R}\}$ and $\partial\Omega$ (see Figure 5). Consider the map

$$h_t : y \in \mathcal{H}_{w^+}(x) \longmapsto y_t = \pi\varphi^t(y, [yw^+]) = (yw^+) \cap \mathcal{H}_{w^+}(x_t).$$

Figure 4. A good chart at $w = (x, [\xi])$

We see that z_t is given by

$$z_t = dh_t(z) = \frac{|x_t w^+|}{|x w^+|} z.$$

This gives the result. $\square$

We have a similar result for $Z^+ \in E^+(w)$. If $Z^+ \in E^+(w)$, with the same notation, we have

$$\|d\varphi^t(Z^+)\|^{\pm} = F(z_t) = \frac{|z|}{2|xw^-|} \left(\frac{|x_t w^-|}{|x_t z_t^+|} + \frac{|x_t w^-|}{|x_t z_t^-|} \right). \tag{4.1}$$

The strict convexity of the convex set and the $\mathcal{C}^1$ regularity of its boundary now yield the following

Corollary 4.5. *Let $Z^- \in E^-$, $Z^+ \in E^+$. The map $t \longmapsto \|d\varphi^t Z^-\|^{\pm}$ is a strictly decreasing bijection from $\mathbb{R}$ onto $(0, +\infty)$, and the map $t \longmapsto \|d\varphi^t Z^+\|^{\pm}$ is a strictly increasing bijection from $\mathbb{R}$ onto $(0, +\infty)$.*
In particular, $W^-(w)$ and $W^+(w)$ are the stable and unstable sets of w for the metric $\|\cdot\|^{\pm}$.

We will prove in the sequel that this metric actually coincides with the metric $\|\cdot\|$.

4.2.2 Identification of stable and unstable bundles for the metric $\|\cdot\|$

For a Riemannian or regular Finsler manifold M of variable negative curva-

Figure 5. Contraction of stable vectors

ture, the construction of stable and unstable manifolds is achieved through the understanding of Jacobi fields, via the following fact: for any $Z_w \in T_w HM$, the vector field Z defined along the orbit of w by $Z(ph^t w) = d\varphi^t Z_w$ is a Jacobi field.
This observation stays valid in Hilbert geometry and, as we have seen, the behaviour of Jacobi fields is easy to understand via parallel transport. In this way, we can identify the stable and unstable bundles in terms of the differential objects of Section 3. They are given by

$$E^u = \{Y + J^F(Y), Y \in VHM\}, E^s = \{Y - J^F(Y), Y \in VHM\} = J^F(E^u),$$

as asserts the next

Proposition 4.6. *Let $Z^s \in E^s$, $Z^u \in E^u$. The map $t \longmapsto \|d\varphi^t Z^s\|$ is a strictly decreasing bijection from $\mathbb{R}$ onto $(0, +\infty)$, and the map $t \longmapsto \|d\varphi^t Z^u\|$ is a strictly increasing bijection from $\mathbb{R}$ onto $(0, +\infty)$.*

To prove this proposition, we need some preparatory observations.

We have $THM = \mathbb{R} \cdot X \oplus E^u \oplus E^s$ and this decomposition is φ^t-invariant. As said above, for any $Z_w \in T_w HM$, the vector field Z defined along $\varphi \cdot w$ by $Z(\varphi^t w) = d\varphi^t Z_w$ is a Jacobi field. Furthermore, if $Z^u_w \in E^u(w)$, we have $D^X(Z^u)(w) = Z^u_w$, and Proposition 3.9 implies that

$$d\varphi^t(Z^u) = e^t T^t Z^u. \tag{4.2}$$

If $Z^s_w \in E^s(w)$, we have $D^X(Z^s)(w) = -Z^s_w$ and

$$d\varphi^t(Z^s) = e^{-t}T^t Z^s. \tag{4.3}$$

To understand the behaviour of $\|d\varphi^t(Z^u)\|$ and $\|d\varphi^t(Z^s)\|$, $Z^u \in E^u$, $Z^s \in E^s$, we have to understand the behaviour of $\|T^t Z^s\|$ and $\|T^t Z^u\|$. Here is the big difference with the regular Finsler or Riemannian case. In these last cases, the parallel transport preserves the Sasaki metric. For the Hilbert geometries under consideration, the parallel transport does not preserve the metric $\|\cdot\|$. In fact, we could prove the

Proposition 4.7. *Let (Ω, d_Ω) be a Hilbert geometry defined by a strictly convex set with $\mathcal{C}^1$ boundary. The following propositions are equivalent:*

- *the boundary of the convex set is $\mathcal{C}^2$ with definite positive Hessian;*
- *for any $w \in H\Omega$, there exists $C > 0$ such that*

$$C^{-1}\|Z\| \leqslant \|T^t Z\| \leqslant C\|Z\|,\ Z \in T_w H\Omega.$$

To understand the parallel transport, we compare it with the parallel transport of the Euclidean structure.
Fix an affine chart and a Euclidean metric $|\cdot|$ on it, in which Ω appears as a bounded open convex set. Denote by $X^e : H\Omega \longrightarrow TH\Omega$ the generator of the Euclidean geodesic flow. Recall that $X = mX^e$.
Let $w \in HM$ and pick a vertical vector $Y(w) \in V_w HM$. Denote by Y and Y^e its parallel transports with respect to X and X^e along $\varphi{\cdot}w$. Let $h = J^F(Y)$ and $h^e = J^{X^e}(Y^e)$ be the corresponding parallel transports of $h(w) = J^F(Y(w))$ and $h^e(w) = J^{X^e}(Y^e(w))$ along $\varphi \cdot w$. The main result is the following

Lemma 4.8. *Along the orbit $\varphi \cdot w$, we have*

$$Y = \left(\frac{m(w)}{m}\right)^{1/2} Y^e.$$

and

$$h = -L_Y m\ X^e + (m(w)m)^{1/2}\ h^e - \frac{m(w)}{m}\ L_{X^e} m\ Y^e.$$

Now, we can complete a

Proof of Proposition 4.6. Consider the vector $Z(w) = Y(w) + h(w) = Y(w) - J^F(Y(w)) \in E^s(w)$. We use the notation of Proposition 4.4 and its proof. In a good chart at w, $L_Y m = 0$ along the orbit $\varphi \cdot w$, hence

$$\|T^t Z(w)\| = F(d\pi(T^t h(w))) = (m(w)m(\varphi^t(w)))^{1/2} F(d\pi(h^e(\varphi^t(w))),$$

where h^e is as above. From Corollary 4.1 and the fact that $h^F H\Omega + VH\Omega = E^s + E^u$, the vector $d\pi(T^t(h(w)))$ is in $T_{x_t}\mathcal{H}_{\varphi^t(w)}$. The Euclidean parallel transport preserves the Euclidean metric so one have $|d\pi(h^e(\varphi^t(w))| = |d\pi(h^e(w))| = |d\pi(h(w))|$. Keeping the same notation (see Figure 6), these two observations give

$$\begin{aligned} \|T^t Z(w)\| &= (m(w)m(\varphi^t(w)))^{1/2}\frac{|d\pi(h(w))|}{2}\left(\frac{1}{|x_t z_t^+|}+\frac{1}{|x_t z_t^-|}\right) \\ &= C(w)\left(\frac{|x_t w^+|^{1/2}}{|x_t z_t^+|}+\frac{|x_t w^+|^{1/2}}{|x_t z_t^-|}\right), \end{aligned}$$

for some constant $C(w) > 0$. This equality is similar to the one in Lemma 4.4. The strict convexity of the convex set and the $\mathcal{C}^1$ regularity of its boundary conclude the proof. □

Figure 6. Action of the parallel transport

4.2.3 Both constructions coincide We will now conclude the proof of Theorem 4.3, through the

Proposition 4.9. *Let (Ω, d_Ω) a Hilbert geometry defined by a strictly convex set with $\mathcal{C}^1$ boundary. We have $E^s = E^-$, $E^u = E^+$ and $\|\cdot\| = \|\cdot\|^{\pm}$.*

I do not know a simple proof of this fact. If the geometry is divisible, then the following lemma concludes:

Lemma 4.10. *If $\|\cdot\|$ and $\|\cdot\|^{\pm}$ are bi-Lipschitz equivalent on $H\Omega$, then $E^s = E^-$, $E^u = E^+$ and $\|\cdot\| = \|\cdot\|^{\pm}$.*

Proof. Pick a vector $Z \in E^-$, decompose it with respect to $E^s \oplus E^u$, and use Corollary 4.5 and Proposition 4.6 to conclude. $\square$

Otherwise, we can do as follows. For related material concerning Benzécri's theorem and Gromov-hyperbolic Hilbert geometries, we should have a look at Section 9 in L. Marquis' contribution.

Proof of Proposition 4.9. Let

$$X = \{(\Omega, x),\ x \in \Omega\}$$

and

$$X' = \{(\Omega, x) \in X,\ \Omega \text{ is strictly convex with } \mathcal{C}^1 \text{ boundary}\}.$$

For $\delta \geqslant 0$, let

$$X^\delta = \{(\Omega, x) \in X,\ \text{the Hilbert geometry } (\Omega, d_\Omega) \text{ is } \delta-\text{hyperbolic}\}.$$

Finally, let

$$X^h = \bigcup_{\delta \geqslant 0} X^\delta$$

be the set of Gromov-hyperbolic Hilbert geometries (see Section 2.3.4 for the definition of Gromov-hyperbolicity).

The space X is equipped with the topology induced by the Gromov-Hausdorff distance on subsets of $\mathbb{RP}^n$ on the Ω-coordinate, and the topology of $\mathbb{RP}^n$ for the x-coordinate. The subsets X', X^h, X^δ inherit the induced topology.

We have $X^\delta \subset X^h \subset X'$ for any $\delta > 0$. The space X^h contains all (Ω, x) such that (Ω, d_Ω) is regular ([18]), and so X^h is dense in X' and X.

J.-P. Benzécri [12] proved that the action of the projective group $\mathrm{PGL}(n+1, \mathbb{R})$ on X is proper and cocompact. Y. Benoist [9] proved that X^δ is closed in X, so the action of $\mathrm{PGL}(n+1, \mathbb{R})$ on X^δ is also proper and cocompact.

Consider the map

$$f : (\Omega, x) \in X' \longrightarrow \mathbb{R}$$

defined by

$$f(\Omega, x) = \max\{\frac{\|Z\|^{\pm}}{\|Z\|},\ Z \in T_w H\Omega,\ \pi(w) = x\}.$$

This map is continuous, positive and $\mathrm{PGL}(n+1, \mathbb{R})$-invariant on X'. So, by the Benoist-Benzécri result, there exists C_δ such that $C_\delta^{-1} \leqslant f \leqslant C_\delta$ on X^δ. Hence, for any Gromov-hyperbolic Hilbert geometry (Ω, d_Ω), $\|\cdot\|$ and $\|\cdot\|^{\pm}$

are bi-Lipschitz equivalent and Lemma 4.10 implies that $E^s = E^-$, $E^u = E^+$ and $\| \cdot \| = \| \cdot \|^{\pm}$. In other words, $f \equiv 1$ on the subset X^h of X'. Since X^h is dense in X' and f is continuous, we have $f \equiv 1$ on X', which concludes the proof. □

5 Hyperbolicity and Lyapunov exponents

In this section, we want to understand the asymptotic behaviour of the geodesic flow locally around a given orbit; in particular, we want to see when it is locally hyperbolic. Since these are local considerations, we work in the universal cover Ω and on $H\Omega$.
In all this part, the Hilbert geometry (Ω, d_Ω) is assumed to be defined by a strictly convex set with $\mathcal{C}^1$ boundary.

5.1 Hyperbolicity of an orbit

The orbit $\varphi \cdot w$ of the point $w \in H\Omega$ of the geodesic flow ends at the point w^+ on the boundary. A fundamental observation is that most of the local properties of the geodesic flow around the orbit $\varphi \cdot w$ are given by the local shape of the boundary $\partial\Omega$ at the point w^+.
Recall the following

Definition 5.1. Let $k \in \mathbb{N}$, $0 < \varepsilon \leqslant 1$ and $\beta \geqslant 2$. Let $f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^k$ be defined on an open subset U. Denote by $T^k f$ its Taylor expansion up to order k, if defined. The function f is said to be

- *of class* $\mathcal{C}^{k+\varepsilon}$ on U if f is $\mathcal{C}^k$ on U and, for some constant $C > 0$,
$$|f(x) - f(y) - T^k f(x)(x-y)| \leqslant C|x-y|^{k+\varepsilon},\ x, y \in U;$$
- *of class* $\mathcal{D}^{k+\varepsilon}$ at $x \in U$ if f is k-times differentiable at x and, for some constant $C > 0$ and a neighborhood V of x,
$$|f(x) - f(y) - T^k f(x)(x-y)| \leqslant C|x-y|^{k+\varepsilon},\ y \in V;$$
- *β-convex* on U if f is $\mathcal{C}^1$ on U and, for some constant $C > 0$,
$$|f(x) - f(y) - T^1 f(x)(x-y)| \geqslant C|x-y|^{\beta},\ x, y \in U.$$
- *β-convex* at $x \in U$ if f is differentiable at x and, for some constant $C > 0$ and a neighborhood V of x,
$$|f(x) - f(y) - T^1 f(x)(x-y)| \geqslant C|x-y|^{\beta},\ y \in V.$$

To study the behaviour around the orbit $\varphi \cdot w$ of the point $w \in H\Omega$, we look at the action of the differential $d\varphi^t$: we pick a vector $Z \in T_w H\Omega$ and look at the asymptotic behaviour of the function $t \longmapsto \|d\varphi^t Z\|$. This function is constant for $Z \in \mathbb{R} \cdot X$. Corollary 4.5 states that, for $Z \in E^s(w)$ (resp. $Z \in E^u(w)$), $\|d\varphi^t Z\|$ decreases to 0 (resp. increases to $+\infty$).
To go further, we want to see when the contraction/expansion on stable and unstable subspaces are exponential.

Definition 5.2. A point $w \in H\Omega$ or its orbit $\varphi \cdot w$ is said to be *hyperbolic* if

$$\sup_{Z^s \in E^s(w)} \limsup_{t \to +\infty} \frac{1}{t} \log \|d\varphi^t Z^s\| < 0,$$

and

$$\inf_{Z^u \in E^u(w)} \liminf_{t \to +\infty} \frac{1}{t} \log \|d\varphi^t Z^u\| > 0.$$

From Equalities (4.2) and (4.3), we see that

$$\limsup_{t \to +\infty} \frac{1}{t} \log \|d\varphi^t Z^s\| = -1 + \limsup_{t \to +\infty} \frac{1}{t} \log \|T^t Z^s\|, \ Z^s \in E^s(w),$$

and

$$\liminf_{t \to +\infty} \frac{1}{t} \log \|d\varphi^t Z^u\| = 1 + \liminf_{t \to +\infty} \frac{1}{t} \log \|T^t Z^u\|, \ Z^u \in E^u(w).$$

Moreover, from the fact that $E^u = J^F(E^s)$ and from the J^F-invariance of the norm, we see that w is hyperbolic if and only if

$$-1 < \liminf_{t \to +\infty} \frac{1}{t} \log \|T^t Z^s\| \leqslant \limsup_{t \to +\infty} \frac{1}{t} \log \|T^t Z^s\| < 1, \ Z^s \in E^s(w).$$

In terms of parallel transport on Ω, w is hyperbolic if and only if

$$-1 < \liminf_{t \to +\infty} \frac{1}{t} \log F(T_w^t v) \leqslant \limsup_{t \to +\infty} \frac{1}{t} \log F(T_w^t v) < 1, \ v \in T_{\pi(w)} \mathcal{H}_{w^+}(\pi(w)).$$

Define

$$\overline{\chi}(w) = \sup_{Z^u \in E^u(w)} \limsup_{t \to +\infty} \frac{1}{t} \log \|d\varphi^t Z^u\|$$

and

$$\underline{\chi}(w) = \inf_{Z^u \in E^u(w)} \liminf_{t \to +\infty} \frac{1}{t} \log \|d\varphi^t Z^u\|.$$

These two numbers control the exponential rate of expansion on E^u along the orbit $\varphi \cdot w$. Using formula (4.1), we can prove the

Proposition 5.3. *An orbit $\varphi \cdot w$ is hyperbolic, with exponents $0 < \underline{\chi} \leq \overline{\chi} < 2$ if and only if the boundary $\partial\Omega$ is of class $\mathcal{C}^{1+\varepsilon}$ and β-convex at the point w^+ for all $1 < 1+\varepsilon < \frac{2}{\overline{\chi}}$ and $\beta \geqslant \frac{2}{\underline{\chi}}$.*

5.2 Regular orbits and Lyapunov exponents

The exponential rate of contraction/expansion can depend on the vector that we consider.

Definition 5.4. A point $w \in H\Omega$ or its orbit $\varphi \cdot w$ is said to be *forward weakly regular* if for any $Z \in T_w H\Omega$, the quantity $\frac{1}{t} \log \|d\varphi^t Z\|$ admits a limit $\chi(Z)$ when t goes to $+\infty$.
It is said to be *weakly regular* if this quantity has the same limit when t goes to $-\infty$.

From previous considerations, to see if a point w is forward weakly regular, we only have to check that the limit exists for all $Z \in E^s(w)$ or that $\eta(v) = \lim_{t\to+\infty} \frac{1}{t} \log F(T_w^t v)$ exists for all $v \in T_{\pi(w)}\mathcal{H}_{w^+}(\pi(w))$.
Let $w \in H\Omega$ be a forward weakly regular point. We call the number $\eta(v)$ the *parallel Lyapunov exponent of the vector v*. These numbers $\{\eta(v),\ v \in T_{\pi(w)}\mathcal{H}_{w^+}(\pi(w))\}$ can take only a finite number $\eta_1(w) < \cdots < \eta_p(w)$ of values, which are called the *parallel Lyapunov exponents of w (or its orbit)*. There is then a decomposition

$$T_{\pi(w)}\mathcal{H}_{w^+}(\pi(w)) = E_1 \oplus \cdots \oplus E_p$$

called *Lyapunov decomposition*, such that, for any vector $v_i \in E_i \smallsetminus \{0\}$,

$$\lim_{t\to+\infty} \frac{1}{t} \log \|T_w^t(v_i)\| = \eta_i(w).$$

I have made a detailed study of (parallel) Lyapunov exponents in [19]. I specify Proposition 5.3 by making a link between the Lyapunov exponents of w and some regularity properties of the boundary $\partial\Omega$ at w^+.

Finally, if we want to look at how the flow transforms volumes along a given orbit $\varphi \cdot w$, we can look at the quantity $\det T_w^t$.

Definition 5.5. A point $w \in H\Omega$ is said to be *regular* if w is weakly regular, with parallel Lyapunov exponents $\eta_1(w) < \cdots < \eta_p(w)$ and Lyapunov decomposition $T_{\pi(w)}\mathcal{H}_{w^+}(\pi(w)) = E_1 \oplus \cdots \oplus E_p$, and if

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \det T_w^t = \sum_{i=1}^{p} \dim E_i\ \eta_i(w) =: \eta(w).$$

Remark that in this definition, the determinant is not computed with respect to a Riemannian metric but with respect to a Finsler metric. This depends on the volume we associate to the Finsler metric. Since all natural volumes are equivalent, let us say that the determinant is computed with respect to the Busemann-Hausdorff volume Vol_Ω.
An important fact is that the Lyapunov exponents of a periodic orbit of the geodesic flow of a quotient manifold $M = \Omega/_\Gamma$ can be explicitly computed. As we have seen (Proposition 2.7), such an orbit corresponds to the conjugacy class of a hyperbolic element $g \in \Gamma$, and we get the following result, which involves the eigenvalues of g:

Proposition 5.6. *Let g be a periodic orbit of the geodesic flow of the manifold $M = \Omega/_\Gamma$, corresponding to a hyperbolic element $g \in \Gamma$. Denote by $\lambda_0 > \lambda_1 > \cdots > \lambda_p > \lambda_{p+1}$ the moduli of the eigenvalues of g. Then*

- *g is regular and has no zero Lyapunov exponent;*
- *the Lyapunov exponents $(\eta_i(g))$ of the parallel transport along g are given by*
$$\eta_i(g) = -1 + 2\ \frac{\log \lambda_0 - \log \lambda_i}{\log \lambda_0 - \log \lambda_{p+1}},\ i = 1 \cdots p;$$
- *the sum of the parallel Lyapunov exponents is given by*
$$\eta(g) = (n+1)\frac{\log \lambda_0 + \log \lambda_{p+1}}{\log \lambda_0 - \log \lambda_{p+1}}.$$

6 Compact quotients

We now want to consider global properties of the geodesic flow of a compact quotient manifold $M = \Omega/_\Gamma$, in the case the convex set Ω is strictly convex (and then with $\mathcal{C}^1$ boundary).

6.1 The Anosov property

Recall the

Definition 6.1. Let $\varphi^t : M \longrightarrow M$ be a $\mathcal{C}^1$ flow on a compact Finsler manifold $(M, \|\cdot\|)$. The flow φ^t is an Anosov flow it there exists a φ^t-invariant decomposition
$$THM = \mathbb{R} \cdot X \oplus F^s \oplus F^u,$$

called the Anosov decomposition, and constants $C, \alpha > 0$ such that for any $t \geqslant 0$,

$$\|d\varphi^t(Z^s)\| \leqslant Ce^{-\alpha t}\|Z^s\|,\ Z^s \in F^s,$$

$$\|d\varphi^{-t}(Z^u))\| \leqslant Ce^{-\alpha t}\|Z^u\|,\ Z^u \in F^u.$$

This property is named under the name of D. V. Anosov. In [2], he proved that the geodesic flow of a negatively curved compact Riemannian manifold satisfies this property and used it to prove its ergodicity relative to its invariant volume (see Section 7.1).

Theorem 6.2 (D. V. Anosov [2], P. Eberlein [23], P. Foulon [26]). *Let F be a regular Finsler metric of negative curvature on a compact manifold M. The geodesic flow of F on HM is an Anosov flow.*

The proof of this result is based on a study of Jacobi fields, whose behaviour can be understood under the curvature assumption. As we have seen, the same can be done in Hilbert geometry, and it is even easier since the curvature is constant: we get actual equalities relating parallel transport and action of the flow.

Theorem 6.3 (Y. Benoist, [10]). *Let $M = \Omega/_{\Gamma}$ be a compact manifold. The geodesic flow of the Hilbert metric is an Anosov flow on HM, with invariant decomposition*

$$THM = \mathbb{R} \cdot X \oplus E^s \oplus E^u.$$

Moreover, the geodesic flow is topologically mixing: for any open subsets A and B of HM, there is a time $T \in \mathbb{R}$ such that for all $t \geqslant T$, $\varphi^t(A) \cap B \neq \emptyset$.

The Anosov property is a direct consequence of Corollary 4.5 together with compactness. We give a short proof of this fact below.
For topological mixing, we refer to Y. Benoist's paper Section 5.3, or to [21] where the proof is extended to some noncompact quotients. The action of the group Γ on the boundary $\partial\Omega$ is minimal, and this allows us to see that the set $\{(x_g^+, x_g^-),\ g \in \Gamma\}$ is dense in $\partial\Omega \times \partial\Omega$, that is periodic orbits of the flow are dense in HM. Topological transitivity[2] follows from this last result. Topological mixing then comes from the non-arithmeticity of the length spectrum, which is a consequence of Theorem 2.6 through the fact that the length of closed orbits is given by the eigenvalues of the elements of the group Γ (see Section 2.4.2).

[2] A flow φ^t is *topologically transitive* for any open subsets A and B of HM, there is a time $T \in \mathbb{R}$ such that $\varphi^T(A) \cap B \neq \emptyset$

Proof of the Anosov property. Choose $T > 0$. The set

$$E_1 = \{Z \in E^s \mid \|Z\| = 1\}$$

of unit stable vectors is compact, and the function $Z \in E_1 \longmapsto \|d\varphi^T(Z)\|$ is continuous, hence attains its maximum for some vector Z_M. Lemma 4.5 tells us that $a := \|d\varphi^T(Z_M)\| < 1$, so that, for all $Z \in E^s$,

$$\|d\varphi^T(Z)\| \leqslant a\|Z\|.$$

Finally, for $t \geqslant 0$, and setting $n = [t/T]$, we get

$$\|d\varphi^t(Z)\| \leqslant a^n \|d\varphi^{t-n} Z\| \leqslant D_T a^n \|Z\| \leqslant \frac{D_T}{a^{t-n}} e^{\log at} \|Z\|,$$

with $D_T = \max\left\{\dfrac{\|d\varphi^t(Z)\|}{\|Z\|},\ Z \in E^s,\ 0 \leqslant t \leqslant T\right\}$.
This gives the upper bound for stable vectors; the same works for unstable ones. $\square$

Notice that there is an important difference with the negatively curved Riemannian case. Here, we do not really know what occurs for small t: we know that the norm of a stable vector decreases but we have no control on the rate of decreasing, unlike in the Riemannian case where even infinitesimal exponential rates are controlled by the bounds on the curvature.

6.2 Regularity of the boundary

As a Corollary of Proposition 5.3, we get the

Corollary 6.4. *Let $M = \Omega/_{\Gamma}$ be a compact quotient manifold. Then the boundary $\partial\Omega$ of Ω is $\mathcal{C}^{1+\varepsilon}$ and β-convex for all*

$$1 < 1 + \varepsilon < \alpha(\Omega) := \frac{2}{\overline{\chi}(HM)},\ \frac{2}{\underline{\chi}(HM)} =: \beta(\Omega) \leq \beta$$

where

$$\overline{\chi}(HM) = \sup_{w \in HM} \overline{\chi}(w), \underline{\chi}(HM) = \inf_{w \in HM} \underline{\chi}(w).$$

In particular, the geodesic flow is $\mathcal{C}^{1+\varepsilon}$ for some $\varepsilon > 0$.

From symmetry arguments, we can see that

$$\frac{1}{\alpha(\Omega)} + \frac{1}{\beta(\Omega)} = 1.$$

Define $\alpha(\Gamma)$ as the biggest $1 < \alpha = 1 + \varepsilon \leqslant 2$ such that $\partial\Omega$ is $\mathcal{C}^{1+\varepsilon}$ at all the fixed points of the elements of Γ. Obviously, we have $\alpha(\Omega) \leq \alpha(\Gamma)$ and O. Guichard gave a geometrical proof of the following:

Theorem 6.5 (O. Guichard [30]). *Let $M = \Omega/\Gamma$ be a compact quotient manifold. We have $\alpha(\Omega) = \alpha(\Gamma)$.*

A dynamical proof had been given by U. Hamenstädt in a more general context. She proved the following which implies the previous result.

Theorem 6.6 (U. Hamenstädt [31]). *Let $M = \Omega/\Gamma$ be a compact quotient manifold. Let*

$$\overline{\chi}(Per) = \sup\{\overline{\chi}(w),\ w \in HM \text{ periodic}\}.$$

We have $\overline{\chi}(Per) = \overline{\chi}(HM)$.

6.3 Geometrical rigidity

The algebraic nature of locally symmetric spaces confer them very specific properties that often turn to be characteristic. Some of these properties directly involve the dynamics of the geodesic flow. For example, let us recall the beautiful

Theorem 6.7 (Y. Benoist-P. Foulon-F. Labourie [11]). *Let (M, g) be a compact Riemannian (or reversible regular Finsler) manifold of negative curvature. If the Anosov decomposition is $\mathcal{C}^\infty$ then (M, g) is locally symmetric.*

Actually, the authors proved a more general result about contact Anosov flows. In particular, they covered the case of a non-reversible Finsler metric: If the Anosov decomposition is $\mathcal{C}^\infty$, then there exists a closed 1-form α on M such that $F' = F + \alpha$ is a locally symmetric Riemannian metric on M.
A similar result, which is easier to prove, allows to characterize the hyperbolic space among all strictly convex divisible Hilbert geometries.

Theorem 6.8 (Y. Benoist [10]). *Let (Ω, d_Ω) be a strictly convex divisible Hilbert geometry. The following are equivalent:*

(1) *the convex set Ω is an ellipsoid;*

(2) *the boundary $\partial\Omega$ is $\mathcal{C}^{1+\varepsilon}$ for all $0 \leqslant \varepsilon < 1$;*

(3) *the Anosov decomposition is $\mathcal{C}^\varepsilon$ for all $0 \leqslant \varepsilon < 1$.*

Proof. The implication (1) $\Rightarrow$ (2) is obvious. (2) $\Leftrightarrow$ (3) comes from the description of stable and unstable bundles as

$$E^u = \{Y + J^F(Y), Y \in VH\Omega\},\ E^s = \{Y - J^F(Y), Y \in VH\Omega\} = J^F(E^u)$$

and the fact that the subbundle $h^F H\Omega$ and the map J^F are $\mathcal{C}^\varepsilon$ if $\partial\Omega$ is $\mathcal{C}^{1+\varepsilon}$. The implication (2) $\Rightarrow$ (1) uses Corollary 6.4. If the boundary $\partial\Omega$ is $\mathcal{C}^{1+\varepsilon}$ for all $1 \leqslant \alpha < 2$, that means $\overline{\chi}(HM) = 1$. In particular, the largest Lyapunov exponent of any periodic orbit is 1. By Proposition 5.6, that means

$$2\,\frac{\log\lambda_0(g) - \log\lambda_i(g)}{\log\lambda_0(g) - \log\lambda_{p+1}(g)} = 1$$

for all $g \in \Gamma$. By Theorem 2.6, this implies Γ is not Zariski-dense in $\mathrm{SL}(n+1,\mathbb{R})$. Theorem 2.5 concludes that Ω is an ellipsoid. □

7 Invariant measures

There are various ways of looking at a dynamical system. In this section, we want to look at geodesic flows from a measure-theoretic point of view.
Let us first recall some basic definitions. We are given a flow φ^t on a topological space X.

- A Borel measure μ on X is *invariant* by the geodesic flow if $\varphi^t * \mu = \mu$, that is, for any Borel subset of X, we have $\mu(\varphi^t(A)) = \mu(A)$.
- An invariant measure is *ergodic* if any invariant Borel subset has zero or full measure.
- An invariant probability measure μ is *mixing* if for any Borel sets A, B, we have $\lim_{t\to+\infty}\mu(\varphi^t(A)\cap B) = \mu(A)\mu(B)$; mixing implies ergodicity.

Geodesic flows of compact negatively curved Riemannian or regular Finsler manifolds preserve lots of probability measures. The simplest are probably those defined by periodic orbits: if w is a periodic point of period $T_w > 0$, we can define an invariant probability measure by pushing forward by the application $t \in [0, T_w] \longmapsto \varphi^t(w)$ the Lebesgue measure of $[0, T_w]$, and normalizing it. If we look at the geodesic flow from this measure-theoretic point of view, the system seems trivial: almost every point is periodic with the same period.

7.1 Absolutely continuous measures

Let M be a manifold and λ the Riemannian measure of an arbitrary Riemannian metric on M. We say that a measure μ on M is *smooth* or is *a volume* if $d\mu = f d\lambda$ with $f : M \longrightarrow \mathbb{R}$ everywhere positive *and continuous*; in the case f is only measurable and nonnegative, we say that μ is an *absolutely continuous* measure.
Since a regular Finsler geodesic flow is in particular a Hamiltonian flow, it preserves the Liouville measure, which is given by $A\wedge dA^{n-1}$, where A denotes the

Hilbert form of the metric (see Section 3.2.3). For a compact negatively curved manifold M, this measure is ergodic and so is the only invariant measure in its Lebesgue class.

7.2 Entropies

7.2.1 Topological entropy Geodesic flows are continuous actions of the group $\mathbb{R}$ by diffeomorphisms. Its topological entropy measures the topological complexity of this action.
Let us recall its definition for a flow $\varphi^t : X \longrightarrow X$ on a compact metric space (X, d). For $t \geqslant 0$, we define the distance d_t on X by :

$$d_t(x, y) = \max_{0 \leqslant s \leqslant t} d(\varphi^s(x), \varphi^s(y)), \; x, y \in X.$$

For any $\varepsilon > 0$ and $t \geqslant 0$, we consider coverings of X by open sets of diameter less than ε for the metric d_t. Let $N(\varphi, t, \varepsilon)$ be the minimal cardinality of such a covering. The *topological entropy* of the flow is then the (well defined) quantity

$$h_{top}(\varphi^t) = \lim_{\varepsilon \to 0} \Big[\limsup_{t \to \infty} \frac{1}{t} \log N(\varphi, t, \varepsilon) \Big].$$

It is important to remark that since the space is compact, the topological entropy does not depend on the metric d, but only on the topology defined by d on the space X: if we replace the metric by a topologically equivalent one, then we will get the same number.

7.2.2 Measure-theoretical entropy To every invariant probability measure m of the flow φ^t is attached its *measure-theoretical entropy* $h_m(\varphi^t)$, which measures the complexity of the flow from this measure point of view. We refer to P. Walters' monograph for a complete description [53]. As could be expected, measures defined by periodic orbits have zero entropy whereas the Liouville volume has always positive entropy (in the negatively curved case). We will give an expression of the entropy of the Liouville measure in a forthcoming section.

7.2.3 Variational principle and measure of maximal entropy A *variational principle* makes a link between measure-theoretical and topological entropies. This principle asserts that

$$h_{top}(\varphi^t) = \sup_{\mu} h_\mu(\varphi^t),$$

where the supremum is taken over all invariant probability measures. A measure which achieves the supremum is called a *measure of maximal entropy.* In some sense, such a measure is well adapted to describe the topological action. The following theorem gives existence and uniqueness of a measure of maximal entropy for hyperbolic dynamics.

Theorem 7.1 (See for instance [36]). *Let W be a compact manifold. A topologically transitive Anosov flow $\varphi^t : W \longrightarrow W$ admits a unique measure of maximal entropy.*

As a consequence, the geodesic flow of a compact negatively curved manifold admits a unique measure of maximal entropy. It is usual to call it the Bowen-Margulis measure because R. Bowen and G. Margulis gave two independent constructions of it; we will denote it by μ_{BM}. Both constructions are of particular interest.
In his PhD thesis, G. Margulis [44, 45] constructed this measure as a product measure. Stable, unstable and orbit foliations provide local coordinate systems $W^s \times W^u \times (-\varepsilon, \varepsilon)$: each point w has a neighborhoord U in which any point is at the intersection of exactly one local stable leaf $W^s \cap U$, one local unstable leaf $W^u \cap U$ and one local orbit $\varphi \cdot w \cap U$. The measure is then described locally as a product $\mu_{BM} = \mu^s \times \mu^u \times dt$, where the measures μ^s and μ^u are measures on stable and unstable leaves, uniquely defined by the following transition property, involving topological entropy:

$$\varphi^t * \mu^s_w = e^{-h_{top}t} \mu^s_{\varphi^t(w)},\ \varphi^t * \mu^u_w = e^{h_{top}t} \mu^u_{\varphi^t(w)},\ w \in W.$$

In [15, 16], R. Bowen proved that for the geodesic flow of a compact hyperbolic manifold, closed orbits were uniformly distributed with respect to the Liouville measure, which in this case is also the measure of maximal entropy. Bowen's construction extends to the case of a topologically transitive Anosov flow, and finally, we find that closed orbits are uniformly distributed with respect to a specific measure, which actually coincides with the one Margulis introduced.

7.3 Harmonic measure and measure of maximal entropy

We have seen (Theorem 6.7) that locally symmetric spaces can be characterized among other Riemannian negatively curved compact manifolds by the regularity of their Anosov decomposition.
We expect them to be also characterized by some property of their invariant measures. Of particular interest are the Liouville measure, the Bowen-Margulis measure and the *harmonic measure*. These three measures are naturally related to various aspects of the manifold:

- the Liouville measure is directly defined through the local Euclidean structure defined by the Riemannian metric;

- the Bowen-Margulis measure describes the distribution of closed orbits of the geodesic flow, hence it is related to the distribution of closed geodesics on the manifold;
- the harmonic measure is related to the Brownian motion on the universal covering of the manifold, hence to the Laplace-Beltrami operator.

For locally symmetric manifolds, such as hyperbolic manifolds, these three measures coincide and there are deep relations between the Riemannian structure, the distribution of closed geodesics and the Laplace-Beltrami operator.
For non-locally symmetric manifolds, we expect the three measures to be distinct, and the previous links to fail. This has been confirmed for the harmonic measure:

Theorem 7.2. *Let M be a compact negatively curved Riemannian manifold.*

- *The Bowen-Margulis measure coincide with the Liouville measure if and only if the universal covering $\tilde{M}$ is asymptotically harmonic, that is, horospheres have constant mean curvature.*
- *The harmonic and Liouville measures coincide if and only if M is locally symmetric.*

The first point is due to F. Ledrappier [38]; he also proved in the same work that Liouville and harmonic measures coincide if and only if the universal covering $\tilde{M}$ is asymptotically harmonic. This last result is used by G. Besson, G. Courtois and S. Gallot [13] together with their minimal entropy rigidity theorem to deduce the second point.
Concerning the Bowen-Margulis measure, the situation had been clarified by A. Katok [35] for surfaces; P. Foulon [28] extended it to the more general case of contact Anosov flows. We give the following version:

Theorem 7.3 (A. Katok [35], P. Foulon [28]). *Let (M, F) be a compact regular Finsler surface of negative curvature. The Liouville measure coincide with the Bowen-Margulis measure if and only if M is a Riemannian manifold of constant curvature.*

7.4 Regular Finsler metrics

For negatively curved regular Finsler metrics, comparing the Liouville measure $A \wedge dA^{n-1}$ with the Bowen-Margulis one is still a relevant question, that has not been considered except for Foulon's contribution [28]. The extension to negatively curved regular Finsler metrics should not change the conjecture: if both measures coincide, the manifold should be Riemannian and locally symmetric.

The case of the harmonic measure is more delicate, because there are various ways to extend the Laplace-Beltrami operator from the Riemannian to the Finsler world, namely because it lives on the manifold and not on its tangent bundle.
There have been various attempts to define such an extension, but they have not been studied further. The more recent is T. Barthelmé's definition [4] which is directly linked to the approach to Finsler geometry we have here. Barthelmé proved in his Ph.D. thesis [5] that the harmonic measure is well defined in the negatively curved case, but its properties have not been studied yet.

7.5 Hilbert geometries

To my knowledge, there is no natural Brownian motion defined on a general Hilbert geometry. All proposed extensions of the Riemannian definition require the regularity of the metric; in particular, there is no natural way of defining a harmonic measure associated to the Hilbert metric of a convex projective compact manifold $M = \Omega/_{\Gamma}$ (apart from the case of hyperbolic manifolds).

Let $M = \Omega/_{\Gamma}$ be a compact manifold, with (Ω, d_Ω) a strictly convex Hilbert geometry, and Γ a discrete subgroup of $\mathrm{Aut}(\Omega)$. We have seen that the geodesic flow on HM is a topologically mixing Anosov flow. In particular, according to Theorem 7.1, it admits a unique measure of maximal entropy μ_{BM}, which is ergodic and mixing. The following general rigidity result implies in particular that this measure is absolutely continuous if and only if M is Riemannian hyperbolic, that is, Ω is an ellipsoid.

Theorem 7.4 (Y. Benoist [10]). *Let $M = \Omega/_{\Gamma}$ be a compact quotient manifold, with (Ω, d_Ω) a strictly convex Hilbert geometry, and Γ a discrete subgroup of $\mathrm{Aut}(\Omega)$. If Ω is not an ellipsoid, the geodesic flow on HM admits no absolutely continuous invariant measure.*

Proof. A. Livšic [41, 42] had already seen that any absolutely continuous measure had to be smooth. In particular, for any periodic point $w \in HM$ of period T, the change of variable formulas implies that

$$\det d_w \varphi^T = 1.$$

This implies in our context (see Section 5) that, for any periodic point $w \in HM$ of period T,

$$\det T_w^T = 1.$$

In the notation of Section 5.2, this implies that, for any $g \in \Gamma$, we have $\eta(g) = 0$. According to Proposition 5.6, this is equivalent to a condition on the eigenvalues of the element of Γ, which cannot be satisfied if Γ is Zariski-dense in $\mathrm{SL}(n+1,\mathbb{R})$ via Theorem 2.6. Theorem 2.5 tells us Ω is an ellipsoid. $\square$

As we have already noticed in Section 3.2.4, the last result can be seen as a consequence of the lack of regularity of the Legendre transform. In particular, this raises the question of characterizing the pullback of the Liouville volume on H^*M by the Legendre transform.

Finally, it seems that there is no counterpart neither for the harmonic measure nor for the Liouville measure in Hilbert geometry. Nevertheless, we will see in the next section that the Liouville measure is also characterized by another property than its absolute continuity, and this allows us to define an extension (in reality two) of the Liouville measure.

8 Entropies

8.1 Volume entropy

The volume entropy of a Hilbert geometry measures the exponential growth of volume of balls. It is defined as

$$h_{vol} = \limsup_{R\to+\infty} \frac{1}{R}\mathrm{Vol}_\Omega(B(o,R))$$

where o is an arbitrary point in Ω. The Hilbert geometry defined by the simplex has zero volume entropy while the volume entropy of the $(n-1)$-dimensional hyperbolic space is $n-1$. We conjecture that those are the extremal cases:

Conjecture. *Let (Ω, d_Ω) be a Hilbert geometry. Then its volume entropy is less than $n-1$.*

The result has been proved in dimension 2 by G. Berck, A. Bernig and C. Vernicos. Recently, Vernicos gave a proof in dimension 3. In higher dimensions, it is known to be true for polytopes, which have zero entropy, and for convex sets whose boundary is $\mathcal{C}^{1+1}$, for which $h_{vol} = n-1$.
For divisible convex sets, we can use a dynamical approach using the following

Theorem 8.1 (A. Manning [43], see also [20]). *Let (Ω, d_Ω) be a Hilbert geometry, $M = \Omega/_\Gamma$ a compact quotient manifold with $G < \mathrm{Isom}(\Omega, d_\Omega)$. The volume entropy of (Ω, d_Ω) equals the topological entropy of the geodesic flow of M.*

8.2 Topological entropy

Theorem 8.2. *Assume $M = \Omega/\Gamma$ is compact, with Ω strictly convex. The topological entropy of the geodesic flow satisfies $h_{top} \leqslant n-1$, with equality if and only if Ω is an ellipsoid.*

Sketch of the proof. Ruelle's inequality [48] implies that

$$h_{top} = h_{\mu_{BM}} \leqslant n - 1 + \int_{HM} \eta \; d\mu_{BM}.$$

Oseledec's Theorem [47] says that the set $\Lambda \subset HM$ of regular points has full μ_{BM} measure. Furthermore, this set is invariant under the flip map $\sigma : HM \longrightarrow HM$ defined by

$$\sigma(x, [\xi]) = (x, [-\xi]),$$

and we have $\eta \circ \sigma = -\eta$ on Λ.
To get the inequality, we just have to remark that, since F is reversible, σ conjugates φ^t and φ^{-t}, hence μ_{BM} is the measure of maximal entropy of both flows and $\sigma * \mu_{BM} = \mu_{BM}$. We conclude that $\int_{HM} \eta \; d\mu_{BM} = 0$.
For the equality case, recall that F. Ledrappier and L.-S. Young proved that equality in Ruelle's inequality

$$h_\mu \leqslant \int \chi^+ \; d\mu$$

occurs if and only if the measure μ has absolutely continuous conditional measures on unstable manifolds. But the diffeomorphism σ sends stable manifolds to unstable ones. Since σ preserves μ_{BM}, the Lebesgue class of its conditional measures on stable and unstable manifolds coincide. In particular, there is equality in Ruelle's inequality for μ_{BM} if and only if μ_{BM} is itself absolutely continuous. By Theorem 7.4, this implies Ω is an ellipsoid. $\square$

The last two theorems imply that Conjecture 8.1 is true for divisible strictly convex Hilbert geometries. The numerous examples of such geometries also provide numerous examples whose volume entropy is positive but strictly smaller than $n-1$.

8.3 Variations of entropy

Given a compact manifold M of dimension d, consider the moduli space $\beta(M)$ of marked convex projective structures on M. Such a structure can be described by a pair (dev, ρ) consisting of

- a *developing map* $dev : \tilde{M} \longrightarrow \mathbb{RP}^n$ which is a diffeomorphism onto a convex set Ω;

- a faithful morphism $\rho : \pi_1(M) \longrightarrow \mathrm{PSL}(n+1,\mathbb{R})$ called the *holonomy map* with respect to which *dev* is $\pi_1(M)$-equivariant; its image $\Gamma = \rho(\pi_1(M))$ divides Ω with quotient $\Omega/_\Gamma$ diffeomorphic to M.

The space $\beta(M)$ is endowed with the topology given by uniform convergence on compact subsets on the first coordinate and the compact-open topology on the second.

Proposition 8.3. *Let M be a compact manifold of dimension n. The entropy function $h_{vol} : \beta(M) \longrightarrow (0, n-1]$ is continuous.*

Proof. Consider a deformation $(\rho_\lambda, dev_\lambda)$, $\lambda \in [-1,1]$ of a given structure (ρ_0, dev_0). These structures provide Finsler metrics F_λ on the abstract manifold M. These metrics vary continuously with λ in the following sense:

$$\lim_{\lambda\to 0} \sup_{TM\smallsetminus\{0\}} \frac{F_\lambda}{F_0} = 1.$$

For let T^1M the unit tangent bundle for F_0. Since T^1M is compact and $\lambda \mapsto dev_\lambda$ is continuous,

$$\lim_{\lambda\to 0} \sup_{T^1M} |F_\lambda - F_0| = 0.$$

Moreover $\min_{T^1M} F_0 > 0$, hence

$$\lim_{\lambda\to 0} \sup_{T^1M} |\frac{F_\lambda}{F_0} - 1| = 0.$$

Homogeneity gives the result, that is there exist reals $C_\lambda \geqslant 1$ such that $\lim_{\lambda\to 0} C_\lambda = 1$ and

$$C_\lambda^{-1} \leqslant \sup_{TM\smallsetminus\{0\}} \frac{F_\lambda}{F_0} \leqslant C_\lambda.$$

Denote by $\tilde{d}_\lambda$ the associated distances on $\tilde{M}$. Let $x, y \in \tilde{M}$, and c_λ be the geodesic from x to y for the metric $\tilde{d}_\lambda$, such that $\int \tilde{F}_\lambda(c'_\lambda(t))\ dt = \tilde{d}_\lambda(x,y)$. Then

$$C_\lambda^{-1} \leqslant \frac{\int \tilde{F}_\lambda(c'_\lambda(t))\ dt}{\int \tilde{F}_0(c'_\lambda(t))\ dt} \leqslant \frac{\tilde{d}_\lambda(x,y)}{\tilde{d}_0(x,y)} \leqslant \frac{\int \tilde{F}_\lambda(c'_0(t))\ dt}{\int \tilde{F}_0(c'_0(t))\ dt} \leqslant C_\lambda.$$

Thus for any $x, y \in \tilde{M}$,

$$C_\lambda^{-1} \leqslant \frac{\tilde{d}_\lambda(x,y)}{\tilde{d}_0(x,y)} \leqslant C_\lambda.$$

From that we clearly get $\tilde{B}_\lambda(x,R) \subset \tilde{B}_0(x, C_\lambda R)$. Hence

$$\begin{aligned} h_{vol}(\rho_\lambda, dev_\lambda) &= \limsup_{R\to\infty} \frac{1}{R}\mathrm{card}\{g \in \pi_1(M),\ gx \in \tilde{B}_\lambda(x,R)\} \\ &\leqslant \limsup_{R\to\infty} \frac{1}{R}\mathrm{card}\{g \in \pi_1(M),\ gx \in \tilde{B}_0(x,C_\lambda R)\} \\ &= C_\lambda h_{vol}(\rho_0, dev_0). \end{aligned}$$

Similarly, $C_\lambda^{-1} h_{vol}(\rho_0, dev_0) \leqslant h_{vol}(\rho_\lambda, dev_\lambda)$. This gives the continuity. $\square$

For a 2-dimensional manifold M of genus $g = 1$, the convex structure is necessarily given by a triangle, whose entropy is zero. If $g \geqslant 2$, the space $\beta(M)$ has been well described by W. Goldman [29] as a manifold diffeomorphic to $\mathbb{R}^{16g-16}$.
X. Nie studied in [46] a special kind of projective structures given by simplicial Coxeter groups, and among other things he showed the following

Theorem 8.4 (X. Nie [46]). *Let M be a compact 2-dimensional manifold. The entropy function $h_{vol} : \beta(M) \longrightarrow (0,1]$ is a surjective continuous map.*

8.4 Sinai-Ruelle-Bowen measures

Consider a flow φ^t on a compact manifold W preserving a smooth probability measure λ. Birkhoff's ergodic theorem asserts that there is set $A_\lambda \subset M$ of full λ-measure on which the quantity

$$\frac{1}{t}\int_0^t g(\varphi^t(x))\ dt$$

converges to some $\overline{g}(x)$ as t goes to $+\infty$ for any integrable Borel function g. This limit function $x \longrightarrow \overline{g}(x)$ is integrable and invariant under the flow. In particular, in the case λ is ergodic, $\overline{g}$ is (λ-almost everywhere) constant equal to $\int g\ d\lambda$.

In general, a flow φ^t on a compact manifold W does not preserve a smooth measure but it seems legitimate to ask ourselves about the behaviour of a random orbit, chosen randomly with respect to some Lebesgue probability measure λ. In particular, does there exist a subset of full λ-measure on which the Birkhoff averages

$$\frac{1}{t}\int_0^t g(\varphi^t(x))\ dt$$

converge for all continuous function g ? If it is the case and the limit is constant (λ-almost everywhere), we can then define a measure μ^+ by

$$\int g \, d\mu^+ = \lim_{t \to +\infty} \frac{1}{t} \int_0^t g(\varphi^t(x)) \, dt.$$

Such a measure is called a *physical measure* for the flow φ^t since it describes the asymptotic distribution of Lebesgue almost every orbit.

Theorem 8.5 (R. Bowen-D. Ruelle [17]). *Any $\mathcal{C}^{1+\varepsilon}$ Anosov flow on a compact manifold M admits a unique physical measure.*

Bowen and Ruelle constructed this measure as the equilibrium measure of the potential

$$f^+ = -\frac{d}{dt}|_{t=0} \log \det d\varphi^t_{E^u}.$$

That is, μ^+ is the unique measure which maximizes the quantity

$$P(f^+, \mu) = h_\mu + \int f^+ \, d\mu$$

among all invariant probability measures. In particular, from general results, μ^+ is ergodic. Furthermore, the pressure $P(f^+) = \sup_\mu P(f^+, \mu)$ is zero and we have

$$h_\mu = -\int f^+ \, d\mu.$$

This implies that μ^+ achieves the equality in Ruelle's inequality and is the unique measure to do so. In particular, from Ledrappier-Young theorem [39], μ^+ is the only invariant probably measure to have absolutely continuous conditional measures on unstable manifolds.

Recall that the flip map $\sigma : HM \longrightarrow HM$ is defined by $\sigma(x, [\xi]) = (x, [-\xi])$. The measure $\mu^- := \sigma * \mu^+$ is also invariant by φ^t and is in fact the physical measure of φ^{-t}. In other words, the Birkhoff averages

$$\frac{1}{|t|} \int_t^0 g(\varphi^t(x)) \, dt$$

converge to $\int g \, d\mu^-$ when t goes to $-\infty$. Similarly, we find that μ^- is the equilibrium measure of the potential

$$f^- = -\frac{d}{dt}|_{t=0} \log \det d\varphi^t_{E^s},$$

and that it has absolutely continuous conditional measures on stable manifolds.

Finally, both measures μ^+ and μ^- coincide if and only if one of them is absolutely continuous.

All of this applies to the geodesic flow of a compact quotient of a strictly convex Hilbert geometry, which is $\mathcal{C}^{1+\varepsilon}$ from Corollary 6.4.

Corollary 8.6. *Let $M = \Omega/_{\Gamma}$ be a compact quotient manifold of a strictly convex Hilbert geometry. Unless Ω is an ellipsoid, the three measures μ_{BM}, μ^+ and μ^- are mutually singular.*

As a direct application of the fact that the measure μ^+ achieves the equality in Ruelle's inequality, we can give a lower bound on its entropy, hence on topological entropy:

Proposition 8.7 ([20]). *Let (Ω, d_Ω) be a strictly convex divisible Hilbert geometry where Ω is not an ellipsoid. Assume $\partial\Omega$ is β-convex for a $\beta \in (2, +\infty)$. Then*

$$h_{top} > \frac{2}{\beta}(n-1).$$

Another application is given in the next section.

8.5 Curvature of the boundary of a divisible convex set

Let us begin with an old theorem of A. D. Alexandrov [1] about convex functions:

Theorem 8.8. *Let $f : U \subset \mathbb{R}^n \longmapsto \mathbb{R}$ be a convex function defined on a convex open set U of $\mathbb{R}^n$. The Hessian matrix $Hess(f) = \left(\frac{\partial^2 f}{\partial_i \partial_j}\right)_{ij}$ exists Lebesgue almost everywhere in U.*

Let Ω be a bounded convex set of the Euclidean space $\mathbb{R}^n$. It is then possible to compute the Hessian of its boundary at Lebesgue almost every point $x \in \partial\Omega$. We will call a $\mathcal{D}^2$ *point* a point x where this is possible.
The Hessian is a positive symmetric bilinear form on the tangent space $T_x\partial\Omega$. It represents the curvature of the boundary at x. When it is degenerate, that means the curvature of the boundary is zero in some tangent direction.
The Hessian is a Euclidean notion, but its degeneracy is not. Namely, if Ω is a properly convex open set of $\mathbb{RP}^n$ and x a point of $\partial\Omega$, we can choose an affine chart centered at x and a metric on it and compute the Hessian of $\partial\Omega$ at x; its degeneracy does not depend on the choice of the affine chart and the metric.
We can *measure* the vanishing of the curvature of $\partial\Omega$ in the following way. Fix

a smooth measure λ^* on the boundary of the dual convex set Ω^*, and call λ its pull-back to $\partial\Omega$. Then λ can be seen as a measure of the curvature of $\partial\Omega$. It can be decomposed as

$$\lambda = \lambda^{ac} + \lambda^{sing},$$

where λ^{ac} is an absolutely continuous measure and λ^{sing} is singular with respect to any Lebesgue measure on $\partial\Omega$. For example, if $\partial\Omega$ is not differentiable at some point x then λ will have an atom at x. The support of λ^{ac} is the closure of the set of $\mathcal{D}^2$ points with nondegenerate Hessian.
Though Ω is convex, it may happen that $\lambda^{ac} = 0$, that is, λ is singular with respect to some (hence any) smooth measure on $\partial\Omega$. This is equivalent to the fact that the Hessian is degenerate at *almost all* $\mathcal{D}^2$ point of $\partial\Omega$. We then say that *the curvature of the boundary is supported on a set of zero Lebesgue measure.*
The following lemma gives a criterion for this to happen, which is due to J.-P. Benzécri [12]:

Lemma 8.9. *Let X_n denote the set of properly convex open sets of $\mathbb{RP}^n$, and pick $\Omega \in X_n$. If there exists a $\mathcal{D}^2$ point $x \in \partial\Omega$ with nondegenerate Hessian, then the closure of the orbit $\mathrm{PGL}(n+1,\mathbb{R}) \cdot \Omega$ in X_n contains an ellipsoid.*

Proof. Choose an affine chart and a Euclidean metric on it such that Ω appears as a bounded convex open set of $\mathbb{R}^n$. Let x be a point of $\partial\Omega$ with nondegenerate Hessian. Let $\mathcal{E}$ be the osculating ball of $\partial\Omega$ at x. It defines a hyperbolic geometry $(\mathcal{E}, d_{\mathcal{E}})$. Pick a point $y \in \partial\mathcal{E}$ distinct from x, and choose a hyperbolic isometry g of $\mathcal{E}$ whose attracting fixed point y and repulsive one x. Now, since $\partial\mathcal{E}$ and $\partial\Omega$ are tangent up to order 2, it is not difficult to see that $g^n \cdot \Omega$ converges to $\mathcal{E}$ when n goes to $+\infty$. This proves the statement. □

As a consequence, we get the following

Theorem 8.10. *Let (Ω, d_Ω) be a divisible Hilbert geometry, and assume Ω is not an ellipsoid. Then any $\mathcal{D}^2$ point has degenerate Hessian. In particular, the curvature of $\partial\Omega$ is supported on a subset of zero Lebesgue measure.*

Proof. This is a consequence of the last proposition and of the following fact: if Ω is divisible then the orbit of Ω under $\mathrm{PGL}(n+1,\mathbb{R})$ is closed in X_n (for a proof of this fact, see Proposition 9.18 in L. Marquis' contribution). □

The boundary of a divisible convex set has thus a quite mysterious geometry. In the case of a strictly convex divisible set, we can be more precise using the fact that the unstable conditional measures of μ^+ are absolutely continuous:

Proposition 8.11 ([19]). *Let* (Ω, d_Ω) *be a* strictly convex *divisible Hilbert geometry. There exists* $\varepsilon > 0$ *such that, for Lebesgue almost every point* $x \in \partial\Omega$*, there exists a* 2*-dimensional subspace* P_x *intersecting* Ω *and containing* x *such that the boundary of the* 2*-dimensional convex set* $\Omega \cap P_x$ *is* $\mathcal{D}^{2+\varepsilon}$ *at* x*. In particular, if* $\Omega \subset \mathbb{RP}^2$*, there exists* $\varepsilon > 0$ *for which its boundary* $\partial\Omega$ *is* $\mathcal{D}^{2+\varepsilon}$ *at Lebesgue almost every point.*

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