LYAPUNOV EXPONENTS IN HILBERT GEOMETRY

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Abstract. We study the Lyapunov exponents of the geodesic flow of a Hilbert geometry. We prove that all the information is contained in the shape of the boundary at the endpoint of the chosen orbit. We have to introduce a regularity property of convex functions to make this link precise. As a consequence, Lyapunov manifolds tangent to the Lyapunov splitting appear very easily. All of this work can be seen as a consequence of convexity and the flatness of Hilbert geometries.

1. Introduction

1.1. Context. A Hilbert geometry is a metric space (Ω, d_Ω) where

- Ω is a *properly convex open set* of the real projective space $\mathbb{RP}^n$, $n \geqslant 2$; *properly* means that there exists a projective hyperplane which does not intersect the closure of Ω, or, equivalently, that there is an affine chart in which Ω appears as a relatively compact set;
- d_Ω is the distance on Ω defined, for two distinct points x, y, by

$$d_\Omega(x,y) = \frac{1}{2}|\log[a,b,x,y]|,$$

where a and b are the intersection points of the line (xy) with the boundary $\partial\Omega$ and $[a,b,x,y]$ denotes the cross ratio of the four points : if we identify the line (xy) with $\mathbb{R}\cup\{\infty\}$, it is defined by $[a,b,x,y] = \frac{|ax|/|bx|}{|ay|/|by|}$.

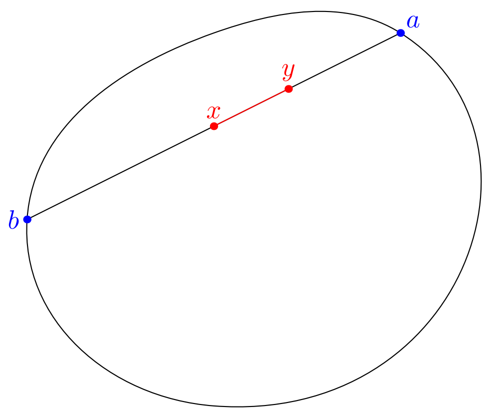

Figure 1. The Hilbert distance

These geometries had been introduced by Hilbert at the end of the nineteenth century as examples of spaces where lines are geodesics, which one can see as a motivation for the fourth of his famous problems: roughly speaking, this problem consisted in finding all geometries for which lines are geodesics.
When Ω is an ellipsoid, one recovers in this way the Beltrami model of the hyperbolic space. This is the only case where a Hilbert geometry is Riemannian. Otherwise, it is only a Finsler space: The Hilbert metric d_Ω is generated by a field of norms F on Ω, the norm $F(x,\xi)$ of a tangent vector $\xi \in T_x\Omega$ being given by the formula

$$F(x,\xi) = \frac{|\xi|}{2}\left(\frac{1}{|xx^+|} + \frac{1}{|xx^-|}\right),$$

where $|\ .\ |$ is an arbitrary Euclidean metric, and x^+ and x^- are the intersection points of the line $x + \mathbb{R}.\xi$ with the boundary $\partial\Omega$.

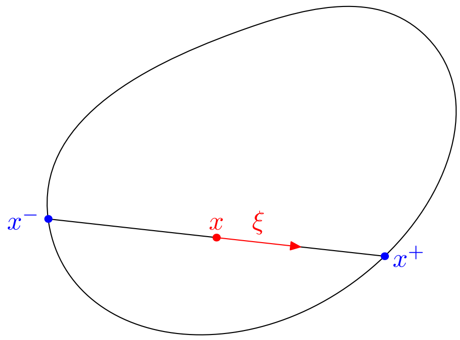

FIGURE 2. The Finsler metric

The geodesic flow φ^t of the Hilbert metric d_Ω is defined on the homogeneous bundle $H\Omega$ of tangent directions by using the fact that lines are geodesics: to find the image of a point $w = (x, [\xi]) \in H\Omega$, consisting of a point and a direction, by φ^t, one follows the geodesic line c_w leaving x in the direction $[\xi]$, and one has $\varphi^t(w) = (c_w(t), [c'_w(t)])$.

The goal of the present article is to understand the infinitesimal behaviour around an orbit going to some point x^+ at infinity in $\partial\Omega$. The main result is that the complete behaviour can be deduced from the shape of the boundary $\partial\Omega$ at the point x^+.

1.2. **Statement of the results.** We make the study in the case Ω is a strictly convex set with $\mathcal{C}^1$ boundary, in which case the flow itself is $\mathcal{C}^1$ and exhibits some hyperbolic behaviour. For example, under this hypothesis, stable and unstable manifolds and distributions can be easily defined by using horospheres (see figure 3); the tangent space $TH\Omega$ then admits a hyperbolic splitting

$$TH\Omega = \mathbb{R}.X \oplus E^s \oplus E^u,$$

into stable and unstable distributions defined by

$$E^s = \{Z \in TH\Omega,\ \lim_{t\to+\infty} \|d\varphi^t(Z)\| = 0\},\ E^u = \{Z \in TH\Omega,\ \lim_{t\to-\infty} \|d\varphi^t(Z)\| = 0\}.$$

Here the norm $\|\ .\ \|$ is a Finsler norm on $H\Omega$ that is naturally related to the Finsler metric on Ω, like the Sasaki metric for Riemannian geodesic flows (see section 2.2).
All of this was already known since the work of Yves Benoist [Ben04], who, among other things, proved that the geodesic flow of a compact quotient of a strictly convex set with $\mathcal{C}^1$ boundary has the Anosov property.

1.2.1. *Lyapunov exponents.* Given a point $w \in H\Omega$ such that $\varphi^{+\infty}(w) = x^+$, we now want to quantify the infinitesimal behaviour of the flow around the orbit $\varphi.w$, by looking at the exponential behaviour of the norms $\|d\varphi^t Z\|$, for $Z \in T_w H\Omega$.

Definition 1. *A point $w \in H\Omega$ is* weakly regular *if, for any $Z \in T_w H\Omega$, the limit*

$$\chi(Z) = \lim_{t\to\pm\infty} \frac{1}{t} \log \|d\varphi^t(Z)\|$$

exists. It is said to be forward *or* backward weakly regular *if the limits exist only when t goes to $+\infty$ or $-\infty$ (or if they differ).*

Given a forward weakly regular point $w \in H\Omega$, the numbers $\chi(Z)$, for any $Z \in T_wH\Omega$, can take only a finite number $\chi_1 < \cdots < \chi_p$ of values, which are called the *Lyapunov exponents* of w. There is then a φ^t-invariant splitting

$$TH\Omega = E_1 \oplus \cdots \oplus E_p$$

along the orbit $\varphi.w$, called *Lyapunov splitting*, such that, for any vector $Z_i \in E_i \smallsetminus \{0\}$,

$$\lim_{t\to+\infty} \frac{1}{t} \log \|d\varphi^t(Z_i)\| = \chi_i.$$

This splitting is actually a subsplitting of the Anosov one, $TH\Omega = \mathbb{R}.X \oplus E^s \oplus E^u$. In fact, E^s and E^u are both isomorphic under projection on the basis to the tangent space to the horosphere, and this provides a symmetry between stable and unstable distributions. We can summarize this by the following

Proposition 1. *Let $w = (x, [\xi]) \in H\Omega$ be a forward weakly regular point and $\mathcal{H}_w$ the horosphere defined by w. Then there are numbers $-1 < \eta_1 < \cdots < \eta_p \leqslant 1$ and a splitting*

$$T_x\mathcal{H}_w = E_1 \oplus \cdots E_p \tag{1.1}$$

such that

- *the Lyapunov splitting*
$$TH\Omega = (\oplus_{i=1}^p E_i^s) \oplus \mathbb{R}.X \oplus (\oplus_{i=1}^p E_i^u)$$
is given by
$$E_i^s = \{Z \in E^s,\ d\pi(Z) \in E_i\},\ E_i^u = \{Z \in E^u,\ d\pi(Z) \in E_i\};$$
- *the associated Lyapunov exponents are given by*
$$\chi_i^s = -1 + \eta_i,\ \chi_i^u = 1 + \eta_i.$$

1.2.2. *Structure of the boundary.* As claimed before, all the information about the asymptotic behaviour of the flow along a specific orbit can be read through the shape of the boundary at the endpoint of this orbit. We need some definitions to explain what this means.

We denote by $\mathtt{Cvx}(n)$ the set of strictly convex $\mathcal{C}^1$ functions $f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $f(0) = f'(0) = 0$, where U is an open convex subset of $\mathbb{R}^n$ containing 0.

Definition 2. *A function $f \in \mathtt{Cvx}(n)$ is* approximately regular *if, for any direction $v \in \mathbb{R}^n \smallsetminus \{0\}$, there is a number $\alpha(v) \in [1, +\infty]$ such that*

$$\lim_{t\to0} \frac{\log \dfrac{f(tv) + f(-tv)}{2}}{\log t} = \alpha(v).$$

The first result about approximate-regularity is the following

Proposition 2. *Let $f \in \mathtt{Cvx}(n)$. The following propositions are equivalent:*

(i) *f is approximately regular;*

(ii) *there exist a splitting $\mathbb{R}^n = \oplus_{i=1}^p G_i$ and numbers $+\infty \geqslant \alpha_1 > \cdots > \alpha_p \geqslant 1$ such that for any $v_i \in G_i \smallsetminus \{0\}$, we have*
$$\lim_{t\to0} \frac{\log \dfrac{f(tv_i) + f(-tv_i)}{2}}{\log t} = \alpha_i;$$

(iii) *there exist a filtration*
$$\{0\} = H_0 \subsetneq H_1 \subsetneq \cdots \subsetneq H_p = \mathbb{R}^n$$
and numbers $+\infty \geqslant \alpha_1 > \cdots > \alpha_p \geqslant 1$, such that, for any $v_i \in H_i \smallsetminus H_{i-1}$, we have
$$\lim_{t\to0} \frac{\log \dfrac{f(tv_i) + f(-tv_i)}{2}}{\log t} = \alpha_i.$$

The numbers α_i are called the Lyapunov exponents *of f.*

It appears that the notion of approximate regularity is projectively invariant so the next definition makes sense:

Definition 3. *The boundary $\partial\Omega$ is approximately regular at the point x^+ if, in some (or any) affine chart and Euclidean metric on it, its graph at x^+ is approximately regular.*

The main theorem is then the following characterization of forward weakly regular points.

Theorem 1. *A point $w \in H\Omega$ is forward weakly regular if and only if $\partial\Omega$ is approximately regular at the endpoint x^+ of the orbit of w. More precisely,*

- *the splitting*
$$T_{x^+}\partial\Omega = G_1 \oplus \cdots \oplus G_p$$
is the projection from x^+ of the splitting (1.1);
- *the Lyapunov exponents $\alpha_1 > \cdots > \alpha_p$ of the boundary at x^+ are given by*
$$\alpha_i = \frac{2}{\eta_i} - 1.$$

The last three results above are actually proved all at the same time, describing from one side the notion of approximate regularity and from another one the behaviour of the geodesic flow. In fact, Proposition 2 and the projective invariance of approximate regularity are consequences of (the proof of) Theorem 1.

1.2.3. *Lyapunov manifolds.* At a consequence of Theorem 1, one gets a decomposition of any horosphere based at an approximately regular point, which is valid for any Hilbert geometry, without hypotheses on the convex set. Of course, this needs an extension of the definitions and the study to a non-regular context, but it is an easy consequence of convexity.

Theorem 2. *Let Ω be a convex proper open subset of $\mathbb{RP}^n$ and fix $o \in \Omega$. Assume $x^+ \in \partial\Omega$ is approximately regular with exponents $+\infty \geqslant \alpha_1 > \cdots > \alpha_p \geqslant 1$ and filtration*
$$\{0\} = H_0 \subsetneq H_1 \subsetneq \cdots \subsetneq H_p = T_{x^+}\partial\Omega.$$
Then the horosphere $\mathcal{H} = \mathcal{H}_{x^+}(o)$ about x^+ passing through o admits a filtration
$$\{o\} = \mathcal{H}_0 \subsetneq \mathcal{H}_1 \subsetneq \cdots \subsetneq \mathcal{H}_p = \mathcal{H},$$
given by $\mathcal{H}_i = \{x \in \mathcal{H} \cap (H_i \oplus \mathbb{R}.ox^+)\}$, $1 \leqslant i \leqslant p$, such that
$$\mathcal{H}_i \smallsetminus \mathcal{H}_{i-1} = \{x \in \mathcal{H} \smallsetminus \{o\},\ \lim_{t\to+\infty} \frac{1}{t} \log d_\Omega(\pi\varphi^t(o,[ox^+]), \pi\varphi^t(x,[xx^+])) = \chi_i^s\},$$
with $\chi_i^s = -2 + \frac{2}{\alpha_i}$, and $\pi : H\Omega \longrightarrow \Omega$ is the projection on the basis.

The filtration of the horosphere in the last theorem implies, via the projection on the basis, a decomposition of the stable and unstable manifolds of a forward regular point into *Lyapunov manifolds*, tangent to the Lyapunov filtration. This is a striking fact that these manifolds appear in a so easy way, only reflecting by projection the shape of the boundary; it can be seen as a consequence of convexity and the flatness of Hilbert geometries.
Recall that in the general theory of nonuniformly hyperbolic systems, the local existence of Lyapunov manifolds is a difficult result whose proof usually uses Hadamard-Perron theorem. Moreover, this general result is valid only at a regular point, that is, a weakly regular point which, furthermore, satisfies
$$\lim_{t\to\pm\infty} \frac{1}{t} \log|\det d\varphi^t| = \sum_{i=1}^{p} \dim E_i\ \chi_i.$$

This assumption is not needed in our context. It is actually equivalent to the graph f of the boundary being approximately regular and satisfying

$$\lim_{t\to 0} \frac{\log \displaystyle\int_{f(u)\leqslant |t|} |u|\ du}{\log t} = \frac{1}{\alpha},$$

with

$$\frac{1}{\alpha} = \sum_{i=1}^{p} \frac{\dim G_i}{\alpha_i}.$$

It then asks the question of knowing if weak regularity would not imply regularity. In other words, is the last property satisfied for all approximately regular functions ?

1.3. **Contents.** The geodesic flow of Hilbert metrics has been studied by Yves Benoist in [Ben04] and by myself in [Cra09]. In the second section of this article, I recall the fundamental results about it.
In the third and fourth parts, we get interested in the Lyapunov exponents of the geodesic flow. These numbers are investigated in section 3, and in section 4, we show that all the information about them is contained in the shape of the boundary at the endpoint of the geodesic ray that had been chosen: this gives rise to Theorem 1. This needs the introduction of *approximate regularity*, whose study requires some time in section 4.
In section 5, we state the main consequences about the asymptotic behaviour of distances when following a geodesic line and explain how to extend it to the nonregular cases: this is Theorem 2. We also show how Lyapunov submanifolds of the geodesic flow appear very naturally in our context.
In he sixth part, some results and questions are raised about the notion of approximate regularity.
In the last part, we give connections with volume entropy, whose study might benefit from the present work.

Unless it is explicitly stated, in particular in sections 5.1 and 7, the convex set Ω is always assumed to be strictly convex with $\mathcal{C}^1$ boundary.

2. The geodesic flow

The homogeneous bundle is the bundle $\pi : H\Omega \longrightarrow \Omega$, with $H\Omega = T\Omega \smallsetminus \{0\}/\mathbb{R}_+^*$, which consists of pairs $(x, [\xi])$, where x is a point of Ω and $[\xi]$ a direction tangent to Ω at x. The geodesic flow $\varphi^t : H\Omega \longrightarrow H\Omega$ of the Hilbert metric is generated by the vector field $X : H\Omega \longrightarrow TH\Omega$. If we choose an affine chart and a Euclidean norm $|\ .\ |$ on it, then X is related to the generator X^e of the Euclidean geodesic flow by $X = mX^e$, where $m : H\Omega \longrightarrow \mathbb{R}$ is defined by

$$m(x, [\xi]) = \frac{2}{\dfrac{1}{|xx^+|} + \dfrac{1}{|xx^-|}},$$

where x^+ and x^- are the intersection points of the line $x + \mathbb{R}.\xi$ with the boundary $\partial\Omega$ (see figure 2). In particular, we see that, under our hypothesis of $\mathcal{C}^1$ regularity of $\partial\Omega$, the function m and the geodesic flow itself are of class $\mathcal{C}^1$.

2.1. **Foulon's dynamical formalism.** This relationship between X and X^e allowed me in [Cra09] to extend the dynamical formalism introduced by Patrick Foulon in [Fou86] (see also the appendix of [Fou92]). This formalism provides objects similar to those used for Riemannian geodesic flows. In particular, it provides a splitting

$$TH\Omega = VH\Omega \oplus h^X H\Omega \oplus \mathbb{R}.X,$$

which is the counterpart of the Levi-Civita connection for Riemannian metrics: $VH\Omega = \ker d\pi$ is the *vertical distribution* which consists of these vectors tangent to the fibers and $h^X H\Omega$ is the *horizontal distribution*, which depends on X. Vertical vectors will be denoted by the letter Y, and horizontal ones by the letter h.
These two distributions are related by the linear operator $J^X : VH\Omega \oplus h^X H\Omega \longrightarrow VH\Omega \oplus h^X H\Omega$ which provides a pseudo-complex structure on $VH\Omega \oplus h^X H\Omega$: J^X satisfies $J^X \circ J^X = -Id$ and exchanges $VH\Omega$ and $h^X H\Omega$.
We also have a *parallel transport* $T^t : TH\Omega \longrightarrow TH\Omega$ along orbits of the flow: for each $w \in H\Omega$, T^t induces a linear map between $T_w H\Omega$ and $T_{\varphi^t(w)} H\Omega$. Furthermore, the parallel transport commutes with J^X and preserves horizontal and vertical distributions.

2.2. **Metric on $H\Omega$.** Dynamical flows are usually studied on Riemannian manifolds, and most of the definitions or theorems are stated in this context. In the case of the geodesic flow of a complete Riemannian manifold M, HM inherits a natural Riemannian metric from the metric on M. In our case, we define a Finsler metric $\| \, . \, \|$ on $H\Omega$, using the splitting $TH\Omega = \mathbb{R}.X \oplus h^X H\Omega \oplus VH\Omega$: if $Z = aX + h + Y$ is some vector of $TH\Omega$, we set

$$\|Z\| = \left(|a|^2 + \frac{1}{2} \left((F(d\pi h))^2 + (F(d\pi J^X(Y)))^2 \right) \right)^{1/2}. \tag{2.1}$$

It allows us to define the length of a $\mathcal{C}^1$ curve $c : [0,1] \to H\Omega$ as

$$l(c) = \int_0^1 \|\dot{c}(t)\| \; dt.$$

It induces a continuous metric $d_{H\Omega}$ on $H\Omega$: the distance between two points $v, w \in H\Omega$ is the minimal length for $\| \, . \, \|$ of a $\mathcal{C}^1$ curve joining v and w.
Remark that, if $\Omega \subset \mathbb{RP}^2$, then $\| \, . \, \|$ is actually a Riemannian metric on $H\Omega$. When Ω is an ellipsoid, we recover the classical Riemannian metric. In any case, $\| \, . \, \|$ is obviously J^X-invariant on $h^X H\Omega \oplus VH\Omega$.
A crucial property is the following lemma which relates the parallel transports with respect to X^e and X. This is (almost) Lemma 4.3 in [Cra09]. (The parallel transport for X^e projects on Ω on the usual Euclidean parallel transport.)

Lemma 2.1. *Let $w \in HM$ and pick a vertical vector $Y(w) \in V_w HM$. Denote by Y and Y^e its parallel transports with respect to X and X^e along the orbit $\varphi.w$. Let $h = J^X(Y)$ and $h^e = J^{X^e}(Y^e)$ be the corresponding parallel transports of $h(w) = J^X(Y(w))$ and $h^e(w) = J^{X^e}(Y^e(w))$ along $\varphi.w$. Then*

$$Y = \left(\frac{m(w)}{m} \right)^{1/2} Y^e$$

and

$$h = -L_Y m \; X^e + (m(w)m)^{1/2} \; h^e - \frac{m(w)}{m} \; L_{X^e} m \; Y^e.$$

2.3. **Horospheres.** *Horospheres* can be defined for *any* Hilbert geometry (Ω, d_Ω). Pick a point $x^+ \in \partial\Omega$. For any point $x \in \Omega$, call $(xx^+) : \mathbb{R} \longrightarrow \Omega$ the geodesic line such that $(xx^+)(0) = x$, $(xx^+)(+\infty) = x^+$. Given a point $x \in \Omega$, there is for each point $y \in \Omega$ a unique time $t_y \in \mathbb{R}$ such that

$$\lim_{t \to +\infty} d_\Omega((xx^+)(t), (yx^+)(t + t_y)) = \inf_{z \in (yx^+)} \left\{ \lim_{t \to +\infty} d_\Omega((xx^+)(t), (zx^+)(t)) \right\}.$$

The horosphere $\mathcal{H}_{x^+}(x)$ through x about x^+ is the set of such "minimal points":

$$\mathcal{H}_{x^+}(x) = \{(yx^+)(t_y),\ y \in \Omega\}.$$

This is a continuous submanifold of Ω.

In the case of a strictly convex set Ω with $\mathcal{C}^1$ boundary, the infimum above is 0, and horospheres can also be defined, as in the hyperbolic space, as level sets of the Busemann functions $b_{x^+}(x,.)$ given by

$$b_{x^+}(x,y) = \lim_{p\to x^+} d_\Omega(x,p) - d_\Omega(y,p).$$

We then have

$$\mathcal{H}_{x^+}(x) = \{y \in \Omega,\ b_{x^+}(x,y) = 0\}. \tag{2.2}$$

A corollary of proposition 3.6 in [Cra09] is the following

Lemma 2.2. *Let $w = (x,[\xi]) \in H\Omega$, $x^\pm = \varphi^{\pm\infty}(w) \in \partial\Omega$ and ξ the unit vector in $[\xi]$. The projection $d\pi(VH\Omega(w) + h^X H\Omega(w))$ is the tangent space at x to both $\mathcal{H}_{x^+}(x)$ and $\mathcal{H}_{x^-}(x)$.*

2.4. **Stable and unstable bundles and manifolds.** The stable and unstable sets of $w \in H\Omega$ are the sets

$$W^s(w) = \{v \in H\Omega,\ \lim_{t\to+\infty} d_{H\Omega}(\varphi^t w, \varphi^t v) = 0\},$$
$$W^u(w) = \{v \in H\Omega,\ \lim_{t\to-\infty} d_{H\Omega}(\varphi^t w, \varphi^t v) = 0\}.$$

In this section we prove the following

Theorem 2.3. *Let $w = (x,[\xi]) \in H\Omega$, $x^\pm = \varphi^{\pm\infty}(w) \in \partial\Omega$. The stable and unstable sets of w are the $\mathcal{C}^1$ submanifolds*

$$W^s(w) = \{v \in H\Omega \mid \varphi^{+\infty}(v) = x^+,\ \pi(v) \in \mathcal{H}_{x^+}(x)\}$$

and

$$W^u(w) = \{v \in H\Omega \mid \varphi^{-\infty}(v) = x^-,\ \pi(v) \in \mathcal{H}_{x^-}(x)\}.$$

Their tangent bundles are given by

$$E^u = \{Y + J^X(Y), Y \in VH\Omega\} \text{ and } E^s = \{Y - J^X(Y), Y \in VH\Omega\} = J^X(E^u).$$

It yields a φ^t-invariant splitting

$$TH\Omega = \mathbb{R}.X \oplus E^s \oplus E^u$$

that we call the Anosov splitting.

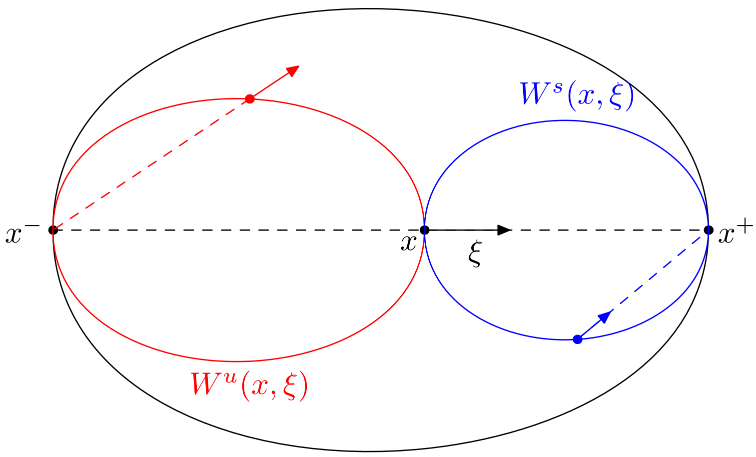

FIGURE 3. Stable and unstable manifolds

Remark 2.4. *To deduce results on (Ω, d_Ω) from results on $(H\Omega, d_{H\Omega})$, it is useful to remark that the projection $\pi : H\Omega \longrightarrow \Omega$ send isometrically stable and unstable manifolds equipped with the metric induced by $\| \,.\, \|$, on horospheres, with the metric induced by d_Ω.*

2.4.1. *A temporary Finsler metric.* Let us set

$$W^-(w) = \{v \in H\Omega \mid \varphi^{+\infty}(v) = x^+,\ \pi(v) \in \mathcal{H}_{x^+}(x)\}$$

and

$$W^+(w) = \{v \in H\Omega \mid \varphi^{-\infty}(v) = x^-,\ \pi(v) \in \mathcal{H}_{x^-}(x)\}.$$

Both sets $W^-(w)$ and $W^+(w)$ are $\mathcal{C}^1$ submanifolds of $H\Omega$. The sets $W^-(w)$, $w \in H\Omega$, foliate $H\Omega$, as well as the sets $W^+(w)$; in general, these foliations are only $\mathcal{C}^0$; they are invariant uner the geodesic flow, that is, $W^-(\varphi^t(w)) = \varphi^t(W^-(w))$ and $W^+(\varphi^t(w)) = \varphi^t(W^+(w))$. We will refer to them as the $-$ and $+$ foliations.

Let E^- and E^+ be the tangent bundles to the $-$ and $+$ foliations. These bundles define a φ^t-invariant splitting of the tangent bundle

$$THM = \mathbb{R}\cdot X \oplus E^- \oplus E^+.$$

Corollary 2.2 implies that $E^- \oplus E^+ = VH\Omega + h^X H\Omega$.

We define a temporary Finsler metric $\| \cdot \|^{\pm}$ by

$$\|Z\|^{\pm} = \left(|a|^2 + F(d\pi Z^+)^2 + F(d\pi Z^-)^2\right)^{1/2}.$$

It is not difficult to see that $W^+(w)$ and $W^-(w)$ are the stable and unstable sets of w for this metric: this is what Corollary 2.6 below asserts.

We first need to make a little computation. To simplify this computation, we will use the projective nature of our objects and choose a good affine chart and a good Euclidean metric on it.

Let $w = (x, [\xi]) \in H\Omega$. *A good chart at w* is an affine chart where the intersection $T_{x^+}\partial\Omega \cap T_{x^-}\partial\Omega$ is contained in the hyperplane at infinity, and a Euclidean structure on it so that the line (xx^+) is orthogonal to $T_{x^+}\partial\Omega$ and $T_{x^-}\partial\Omega$ (see Figure 4).

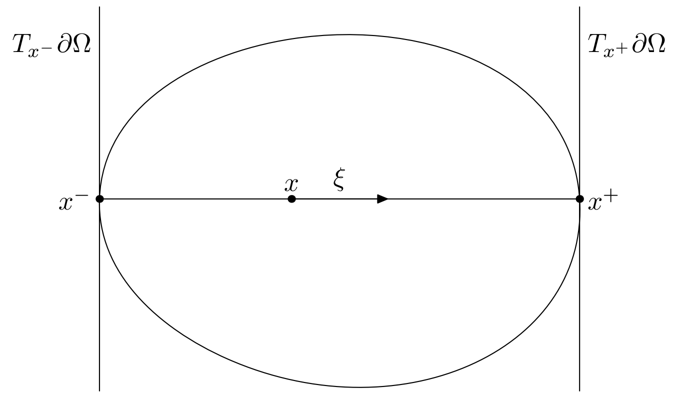

FIGURE 4. A good chart at $w = (x, [\xi])$

Lemma 2.5. *Let $w \in H\Omega$, $Z^- \in E^-(w)$ and fix a good chart at w. Set $x = \pi(w)$, $x_t = \pi\varphi^t(w)$, $z = \pi(Z^-)$, $z_t = d\pi d\varphi^t(Z^-)$. We have*

$$\|d\varphi^t(Z^-)\|^{\pm} = F(z_t) = \frac{|z|}{2|xx^+|}\left(\frac{|x_t x^+|}{|x_t z_t^+|} + \frac{|x_t x^+|}{|x_t z_t^-|}\right).$$

Proof. We have $\|d\varphi^t(Z^-)\|^\pm = F(z_t)$ by definition of the metric $\|\cdot\|^\pm$. Now, by definition of F, we get

$$F(z_t) = \frac{|z_t|}{2}\left(\frac{1}{|x_t z_t^+|} + \frac{1}{|x_t z_t^-|}\right),$$

where z_t^+ and z_t^- are the intersection points of the line $\{x_t + \lambda z_t,\ \lambda \in \mathbb{R}\}$ and $\partial\Omega$ (see Figure 5). Consider the map

$$h_t : y \in \mathcal{H}_{x^+}(x) \longmapsto y_t = \pi\varphi^t(y, [yx^+]) = (yx^+) \cap \mathcal{H}_{x^+}(x_t).$$

We see that z_t is given by

$$z_t = dh_t(z) = \frac{|x_t x^+|}{|xx^+|} z.$$

This gives the result. □

We have a similar result for $Z^+ \in E^+(w)$. If $Z^+ \in E^+(w)$, with the same notation, we have

$$\|d\varphi^t(Z^+)\|^\pm = F(z_t) = \frac{|z|}{2|xx^-|}\left(\frac{|x_t x^-|}{|x_t z_t^+|} + \frac{|x_t x^-|}{|x_t z_t^-|}\right). \tag{2.3}$$

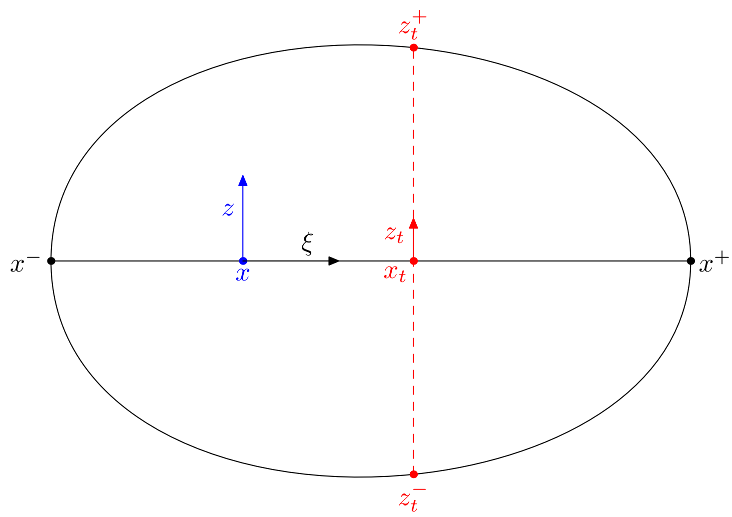

FIGURE 5. Contraction of stable vectors

The strict convexity of the convex set and the $\mathcal{C}^1$ regularity of its boundary now yield the following

Corollary 2.6. *Let $Z^- \in E^-$, $Z^+ \in E^+$. The map $t \longmapsto \|d\varphi^t Z^-\|^\pm$ is a strictly decreasing bijection from $\mathbb{R}$ onto $(0, +\infty)$, and the map $t \longmapsto \|d\varphi^t Z^+\|^\pm$ is a strictly increasing bijection from $\mathbb{R}$ onto $(0, +\infty)$. In particular, $W^-(w)$ and $W^+(w)$ are the stable and unstable sets of w for the metric $\|\cdot\|^\pm$.*

2.4.2. *Identification of stable and unstable bundles for the metric $\|\cdot\|$.* Let us set

$$E^u := \{Y + J^X(Y), Y \in VHM\},\ E^s := \{Y - J^X(Y), Y \in VHM\} = J^X(E^u).$$

These bundles satisfy the following properties:

Lemma 2.7 ([Cra09], Section 4.1 and equation (15))**.** *E^s and E^u are invariant under the geodesic flow and $TH\Omega$ splits as*

$$TH\Omega = \mathbb{R}.X \oplus E^s \oplus E^u.$$

If $Z^s \in E^s$, $Z^u \in E^u$, then

$$d\varphi^t(Z^u) = e^t T^t(Z^u),\ d\varphi^t(Z^s) = e^{-t} T^t(Z^s),$$

where T^t denote the parallel transport introduced in section 2.1. Moreover, the operator J^X exhanges E^u and E^s and

$$d\varphi^t J^X(Z^s) = e^{2t} J^X(d\varphi^t Z^s).$$

Remark that the second equality is just a consequence of the fact that J^X commutes with the parallel transport: we have

$$d\varphi^t J^X(Z^s) = e^t T^t J^X(Z^s) = e^t J^X T^t(Z^s) = e^{2t} J^X(d\varphi^t Z^s).$$

Remark also that, for $Z^s \in E^s,\ Z^u \in E^u$, we have

$$\|Z^s\| = F(d\pi Z^s) \text{ and } \|Z^u\| = F(d\pi Z^u).$$

In general, the $\|\cdot\|$ norm of a vector $Z = aX + Z^s + Z^u$ is given by

$$\|Z\| = \left(|a|^2 + F(d\pi Z^s)^2 + F(d\pi Z^u)^2\right)^{1/2}.$$

The main result, already contained in [Cra09], is the following

Proposition 2.8. *Let $Z^s \in E^s$, $Z^u \in E^u$. The map $t \longmapsto \|d\varphi^t Z^s\|$ is a strictly decreasing bijection from $\mathbb{R}$ onto $(0, +\infty)$, and the map $t \longmapsto \|d\varphi^t Z^u\|$ is a strictly increasing bijection from $\mathbb{R}$ onto $(0, +\infty)$.*

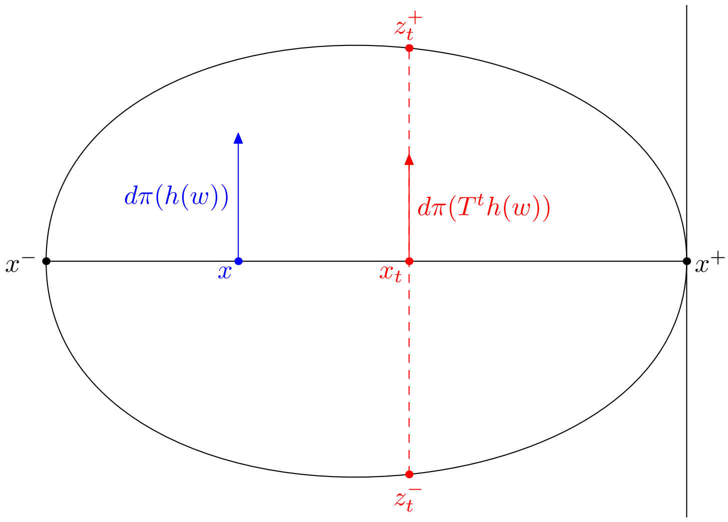

FIGURE 6. Action of the parallel transport

Proof. Consider the vector $Z(w) = Y(w) + h(w) = Y(w) - J^X(Y(w)) \in E^s(w)$. We use the notation of Proposition 2.5 and its proof. In a good chart at w, $L_Y m = 0$ along the orbit $\varphi \cdot w$; hence, from Lemma 2.1, we have

$$\|T^t Z(w)\| = F(d\pi(T^t h(w))) = (m(w)m(\varphi^t(w))^{1/2} F(d\pi(h^e(\varphi^t(w))).$$

From Lemma 2.2 and the fact that $h^X H\Omega + VH\Omega = E^s + E^u$, the vector $d\pi(T^t(h(w)))$ is in $T_{x_t}\mathcal{H}_{\varphi^t(w)}$. The Euclidean parallel transport preserves the Euclidean metric so one have $|d\pi(h^e(\varphi^t(w))| = |d\pi(h^e(w))| = |d\pi(h(w))|$. Keeping the same notation (see Figure 6), these two observations give

$$\begin{aligned}
\|T^t Z(w)\| &= (m(w)m(\varphi^t(w))^{1/2} \frac{|d\pi(h(w))|}{2} \left(\frac{1}{|x_t z_t^+|} + \frac{1}{|x_t z_t^-|} \right) \\
&= \frac{(|xx^+||xx^-||x_t x^-|)^{1/2}}{|x^+x^-|} \frac{|d\pi(h(w))|}{2} \left(\frac{|x_t x^+|^{1/2}}{|x_t z_t^+|} + \frac{|x_t x^+|^{1/2}}{|x_t z_t^-|} \right).
\end{aligned}$$

Lemmas 2.7 above and 2.9 below implies that
$$\|d\varphi^t Z(w)\| = e^{-t}\|T^t Z(w)\| = \frac{|xx^-|}{|x^+x^-|}\frac{|d\pi(h(w))|}{2}\left(\frac{|x_tx^+|}{|x_tz_t^+|} + \frac{|x_tx^+|}{|x_tz_t^-|}\right).$$
This equality is the same as the one in Lemma 2.5. The strict convexity of the convex set and the $\mathcal{C}^1$ regularity of its boundary conclude the proof. □

Lemma 2.9. *We have*
$$\frac{|x_tx^-|}{|x_tx^+|} = e^{2t}\frac{|xx^-|}{|xx^+|}.$$

Proof. We have $d_\Omega(x, x_t) = t$, which implies
$$e^{2t} = \frac{|xx^-|}{|xx^+|}\frac{|x_tx^-|}{|x_tx^+|},$$
and yields the result. □

2.4.3. *Both constructions coincide.* From one side, Corollary 2.6 asserts that W^- and W^+ are the stable and unstable manifolds for the metric $\|\cdot\|$. From another side, by Proposition 2.8, the bundles E^s and E^u should be the tangent spaces to the stable and unstable foliations for the metric $\|\cdot\|$; but there is no general result which ensures their integrability. We will now conclude the proof of Theorem 2.3, through the

Proposition 2.10. *Let (Ω, d_Ω) a Hilbert geometry defined by a strictly convex set with $\mathcal{C}^1$ boundary. We have $E^s = E^-$, $E^u = E^+$ and $\|\cdot\| = \|\cdot\|^\pm$.*

The previous results reduce the problem to proving that $\|\cdot\|$ and $\|\cdot\|^\pm$ are bi-Lipschitz equivalent on $H\Omega$, that is, $C^{-1} \leqslant \frac{\|Z\|}{\|Z\|^\pm} \leqslant C$ for all $Z \in H\Omega$ and some $C \geqslant 1$:

Lemma 2.11. *If $\|\cdot\|$ and $\|\cdot\|^\pm$ are bi-Lipschitz equivalent on $H\Omega$, then $E^s = E^-$, $E^u = E^+$ and $\|\cdot\| = \|\cdot\|^\pm$.*

Proof. Pick a vector $Z \in E^-$, decompose it with respect to $E^s \oplus E^u$, and use Corollary 2.6 and Proposition 2.8 to conclude. □

Now, we can use results of Benzécri [Ben60] and Benoist [Ben03] to conclude. Let
$$\mathcal{X} = \{(\Omega, x),\ x \in \Omega\}$$
and
$$\mathcal{X}' = \{(\Omega, x) \in \mathcal{X},\ \Omega \text{ is strictly convex with } \mathcal{C}^1 \text{ boundary}\}.$$
For $\delta \geqslant 0$, let
$$\mathcal{X}^\delta = \{(\Omega, x) \in \mathcal{X},\ \text{the Hilbert geometry } (\Omega, d_\Omega) \text{ is } \delta-\text{hyperbolic}\}.$$
Finally, let
$$\mathcal{X}^h = \bigcup_{\delta \geqslant 0} \mathcal{X}^\delta$$
be the set of Gromov-hyperbolic Hilbert geometries.

The space $\mathcal{X}$ is equipped with the topology induced by the Gromov-Hausdorff distance on subsets of $\mathbb{RP}^n$ on the Ω-coordinate, and the topology of $\mathbb{RP}^n$ for the x-coordinate. The subsets $\mathcal{X}', \mathcal{X}^h, \mathcal{X}^\delta$ inherit the induced topology.

We have $\mathcal{X}^\delta \subset \mathcal{X}^h \subset \mathcal{X}'$ for any $\delta > 0$. The space $\mathcal{X}^h$ contains all (Ω, x) for which $\partial\Omega$ is $\mathcal{C}^2$ with nondegenerate Hessian ([CP04]); hence $\mathcal{X}^h$ is dense in $\mathcal{X}'$ and $\mathcal{X}$.

Theorem 2.12. *Let $G_n = PGL(n+1, \mathbb{R})$ be the group of projective transformations of $\mathbb{RP}^n$.*

- *The action of G_n on $\mathcal{X}$ is proper and cocompact. (Benzécri* [Ben60]*)*
- *Let $\delta > 0$. The set $\mathcal{X}^\delta$ is closed in $\mathcal{X}$; in particular, the action of G_n on $\mathcal{X}$ is proper and cocompact. (Benoist,* [Ben03]*)*

We can now complete a

Proof of Proposition 2.10. Consider the map

$$f : (\Omega, x) \in \mathcal{X}' \longrightarrow \mathbb{R}$$

defined by

$$f(\Omega, x) = \max\{\frac{\|Z\|^{\pm}}{\|Z\|},\ Z \in T_w H\Omega,\ \pi(w) = x\}.$$

This map is continuous, positive and $PGL(n+1, \mathbb{R})$-invariant on $\mathcal{X}'$. So, by Theorem 2.12, there exists C_δ such that $C_\delta^{-1} \leqslant f \leqslant C_\delta$ on $\mathcal{X}^\delta$. Hence, for any Gromov-hyperbolic Hilbert geometry (Ω, d_Ω), $\|\cdot\|$ and $\|\cdot\|^{\pm}$ are bi-Lipschitz equivalent and Lemma 2.11 implies that $E^s = E^-$, $E^u = E^+$ and $\|\cdot\| = \|\cdot\|^{\pm}$. In other words, $f \equiv 1$ on the subset $\mathcal{X}^h$ of $\mathcal{X}'$. Since $\mathcal{X}^h$ is dense in $\mathcal{X}'$ and f is continuous, we have $f \equiv 1$ on $\mathcal{X}'$, which concludes the proof. □

3. Lyapunov exponents

The goal now is to understand for a given tangent vector $Z \in TH\Omega$ the asymptotic behaviour of the norms $\|d\varphi^t Z\|$ when t goes to $\pm\infty$. In particular, we want to catch some exponential behaviour by looking at the limits, when they exist,

$$\chi^{\pm}(Z) = \lim_{t \to \pm\infty} \frac{1}{t} \log \|d\varphi^t(Z)\|.$$

When $\chi^{\pm}(Z) \neq 0$, this means that $\|d\varphi^t Z\|$ has exponential behaviour when $t \to \pm\infty$: for any $\epsilon > 0$, there exists some $C_\epsilon > 0$ such that, whenever $t > 0$,

$$C_\epsilon^{-1} e^{(\pm\chi^{\pm}(Z) - \epsilon)t} \leqslant \|d\varphi^t(Z)\| \leqslant C_\epsilon e^{(\pm\chi^{\pm}(Z)+\epsilon)t}.$$

3.1. **Regular points and Oseledets' theorem.**

In the introduction, we recall that a point is weakly regular if Lyapunov exponents exist at this point (Definition 1). More generally, we can state the following

Definition 3.1. *Let φ^t be a $\mathcal{C}^1$-flow on a Finsler manifold $(W, \|\ .\ \|)$. A point $w \in W$ is said to be* regular *if there exist a φ^t-invariant splitting*

$$TW = E_1 \oplus \cdots \oplus E_p,$$

along the orbit $\varphi.w$, called Lyapunov splitting*, and real numbers*

$$\chi_1(w) < \cdots < \chi_p(w),$$

called Lyapunov exponents*, such that, for any vector $Z_i \in E_i \smallsetminus \{0\}$,*

$$\lim_{t \to \pm\infty} \frac{1}{t} \log \|d\varphi^t(Z_i)\| = \chi_i(w), \tag{3.1}$$

and

$$\lim_{t \to \pm\infty} \frac{1}{t} \log |\det d\varphi^t| = \sum_{i=1}^{p} \dim E_i\ \chi_i(w). \tag{3.2}$$

The point w is said to be forward *or* backward regular *if this behaviour occurs only when t goes respectively to $+\infty$ or $-\infty$.*

In this definition, we have to precise what is meant by $\det d\varphi^t$, since $\|\ .\ \|$ is not a Riemannian metric. The determinant $\det d\varphi^t$ just measures the effect of φ^t on volumes. But associated to the Finsler metric $\|\ .\ \|$ is the Busemann volume vol_W, which is the volume form defined by saying that, in each tangent space $T_w W$, the volume of the unit ball of $\|\ .\ \|$ is the same as the volume of the Euclidean unit ball of the same dimension. In other words, given an arbitrary Riemannian metric g on W with Riemannian volume vol_g, we have, at the point $w \in W$,

$$dvol_W(w) = \frac{vol_g(B_g(w,1))}{vol_g(B(w,1))} dvol_g(w),$$

where $B(w,1)$ and $B_g(w,1)$ denote the unit balls in T_wW for, respectively, $\| \ . \ \|$ and g. The determinant $\det d\varphi^t$ is then defined in this way: if A is some Borel subset of T_wW with non-zero volume, then

$$|\det d_w\varphi^t| = \frac{vol_W(\varphi^t(w))(d\varphi^t A)}{vol_W(w)(A)}.$$

The essential result about regular points is the following theorem of Oseledets, which, given an invariant probability measure of the flow, gives a condition for almost all points to be regular.

Theorem 3.2 (Osedelets' ergodic multiplicative theorem [Ose68])**.** *Let φ^t be a $\mathcal{C}^1$ flow on a Finsler manifold $(W, \| \ . \ \|)$ and μ a φ^t-invariant probability measure. If*

$$\frac{d}{dt}_{|t=0} \log \|d\varphi^{\pm t}\| \in L^1(W,\mu), \tag{3.3}$$

then the set of regular points has full measure.

Assumption (3.3) means that the flow does not expand or contract locally too fast. This essentially allows us to use Birkhoff's ergodic theorem to prove the theorem.
The next lemma proves that our geodesic flow satisfies assumption (3.3). Obviously, Oseledets' theorem is not interesting on $H\Omega$ itself since there is no finite invariant measure. But it can be applied for any invariant measure of the geodesic flow of a given a quotient manifold $M = \Omega/\Gamma$.

From [Cra09], we know that our Finsler metric is smooth in the direction of the flow, so condition (3.3) makes sense. Furthermore, Oseledets' theorem is usually stated on a Riemannian manifold but it is still valid for a Finsler one: using John's ellipsoid, it is always possible to define a Riemannian metric $\| \ . \ \|_J$ which is bi-Lipschitz equivalent to $\| \ . \ \|$, that is, such that

$$\frac{1}{\sqrt{n}}\|Z\|_J \leqslant \|Z\| \leqslant \sqrt{n}\|Z\|_J, \ Z \in TW,$$

where n is the dimension of the manifold; of course, there is no reason for this metric $\| \ . \ \|_J$ to be even continuous but it is not important.

Lemma 3.3. *For any $Z^s \in E^s, \ Z^u \in E^u$, we have*

$$-2 \leqslant \frac{d}{dt}_{|t=0} \|d\varphi^t Z^s\| \log \leqslant 0 \leqslant \frac{d}{dt}_{|t=0} \|d\varphi^t Z^u\| \leqslant 2.$$

In particular, for any $t \in \mathbb{R}$ and $Z \in TH\Omega$,

$$e^{-2|t|}\|Z\| \leqslant \|d\varphi^t(Z)\| \leqslant e^{2|t|}\|Z\|.$$

This lemma clearly implies the already known fact (coming from Proposition 2.8) that Lyapunov exponents are all between -2 and 2. But it is more precise: it gives that the rate of expansion/contraction is *at any time* between -2 and 2, not only asymptotically, and that is what is essential to apply Oseledets' theorem.

Proof. It is a direct corollary of Proposition 2.8. We know that $t \mapsto \|d\varphi^t Z^s\|$ is decreasing, hence

$$\lim_{t\to 0} \frac{1}{t} \log \|d\varphi^t Z^s\| \leqslant 0.$$

But we also know from Lemma 2.7 that

$$\|d\varphi^t Z^s\| = e^{-2t}\|d\varphi^t J^X(Z^s)\|.$$

Since $J^X(Z^s) \in E^u$, Proposition 2.8 tells us that $t \mapsto \|d\varphi^t J^X(Z^s)\|$ is increasing, hence

$$\lim_{t\to 0} \frac{1}{t} \log \|d\varphi^t J^X(Z^s)\| \geqslant 0$$

and

$$\lim_{t\to 0} \frac{1}{t} \log \|d\varphi^t Z^s\| \geqslant -2.$$

Using J^X, we get the second inequality, and by integrating, we get the last one. □

3.2. **Symmetries.** There are lots of symmetries in our geodesic flow that we should exploit to reduce our study. First, thanks to the Anosov splitting $TH\Omega = \mathbb{R}.X \oplus E^s \oplus E^u$, it is enough to study the asymptotic behaviour of the norms $\|d\varphi^t Z\|$ for $Z \in E^s$ or $Z \in E^u$; of course, $d\varphi^t X = X$, and we can recover the asymptotic behaviour of any vector Z by decomposing it with respect to the Anosov splitting.

Second, via Lemma 2.7, the operator J^X provides a $\|\ .\ \|$-isometry between E^s and E^u. All of this easily implies the following

Proposition 3.4. *Along a forward weakly regular orbit, the Lyapunov splitting can be written as*

$$TH\Omega = (\oplus_{i=1}^p E_i^s) \oplus \mathbb{R}.X \oplus (\oplus_{i=1}^p E_i^u) \tag{3.4}$$

with $E_i^s \subset E^s$, $E_i^u \subset E^u$ and $E_i^u = J^X(E_i^s)$. The corresponding Lyapunov exponents χ_i^s, χ_i^u satisfy

$$\chi_i^s = \chi_i^u - 2,\ 1 \leqslant i \leqslant p.$$

Just remark that in the last statement, the space corresponding to the Lyapunov exponent 0 may be bigger than $\mathbb{R}.X$, containing E_p^s and/or E_1^u.

Finally, thanks to the reversibility of the Hilbert metric, it suffices to study what occurs when t goes to $+\infty$ by using the *flip map* σ defined by

$$\begin{array}{cccc} \sigma : & H\Omega & \longrightarrow & H\Omega \\ & w = (x, [\xi]) & \longmapsto & (x, [-\xi]). \end{array}$$

The following result is not difficult to prove once we know how is defined the operator J^X. (A proof can be found in my Ph.D thesis [Cra11].)

Lemma 3.5. *The differential $d\sigma$ anticommutes with J^X, that is, $J^X \circ d\sigma = -d\sigma \circ J^X$. As a consequence, σ preserves the splitting $TH\Omega = \mathbb{R}.X \oplus h^X H\Omega \oplus VH\Omega$, is a $\|\ .\ \|$-isometry and exchanges stable and unstable distributions and foliations.*

In what follows, we will then focus on forward (weakly) regular points.

3.3. **Parallel Lyapunov exponents.** Consider an unstable vector $Z^u = Y + J^X(Y) \in E^u$ and the corresponding stable one $Z^s = J^X(Z^u) = J^X(Y) - Y \in E^s$. From Lemma 2.7, since J^X commutes with T^t and T^t preserves horizontal and vertical distributions, we have

$$d\varphi^t Z^u = e^t T^t Z^u = e^t (T^t Y - J^X(T^t Y)),\ d\varphi^t Z^s = e^{-t} T^t Z^s = e^{-t}(T^t Y - J^X(T^t Y)).$$

But, from the very definition of the metric $\|\ .\ \|$, we have

$$\|T^t Z^u\| = \|T^t Z^s\| = F(d\pi J^X(T^t Y)) = F(d\pi(T^t h)),$$

where $h = J^X(Y) \in h^X H\Omega$. These quantities $F(d\pi(T^t h))$, when t goes to $+\infty$, are the ones we need to understand.

By Corollary 2.2, given a point $w = (x, [\xi]) \in H\Omega$, the projection of the horizontal space $h^X H\Omega$ at w on $T\Omega$ is precisely the tangent space $T_x \mathcal{H}_w$ to the horosphere $\mathcal{H}_w$ at the point x. We now define a parallel transport along oriented geodesics on Ω that will contain all the information

we need and become the main object of our study.
Let fix a point $x^+ \in \partial\Omega$. Denote by $W^s(x^+) = \{w \in H\Omega,\ \varphi^{+\infty}(w) = x^+\}$ the weak stable manifold associated to x^+, consisting of these points w that end at x^+. Obviously, the map π identifies $W^s(x^+)$ with Ω, and we will call $\pi_{x^+}^{-1}$ the inverse of $\pi_{|_{W^s(x^+)}}$; we have $\pi_{x^+}^{-1}(x) = (x, [xx^+])$. The radial flow $\varphi_{x^+}^t$ is the flow on Ω defined via

$$\varphi_{x^+}^t = \pi\varphi^t\pi_{x^+}^{-1}.$$

It is generated by the vector field X_{x^+} such that $[X_{x^+}] = [xx^+]$ and $F(X_{x^+}) = 1$. Obviously, this flow preserves the set $\{\mathcal{H}_w,\ w \in W^s(x^+)\}$ of horospheres based at x^+, by sending $\mathcal{H}_w$ on $\mathcal{H}_{\varphi^t(w)}$; also it contracts the Hilbert distance d_Ω. Finally, the space $T\Omega$ admits a $\varphi_{x^+}^t$-invariant splitting

$$T\Omega = \mathbb{R}.X_{x^+} \oplus T\mathcal{H}_{x^+},$$

where $T\mathcal{H}_{x^+}$ is the bundle over Ω defined as

$$T\mathcal{H}_{x^+} = \{T_x\mathcal{H}_w,\ w = (x, [\xi]) \in W^s(x^+)\}.$$

Furthermore, from the very definition of the radial flow, we have $d\varphi_{x^+}^t = d\pi d\varphi^t d\pi_{x^+}^{-1}$; so, for any vector $v \in T\mathcal{H}_{x^+}$, we have

$$d\varphi_{x^+}^t(v) = d\pi d\varphi^t d\pi_{x^+}^{-1}(v),$$

where $d\pi_{x^+}^{-1}(v)$ is a stable vector. The action of $d\varphi^t$ on E^s can be deduced from the action of the parallel transport on E^s, and we now define a parallel transport on Ω to get the same kind of relations.

Definition 3.6. *Let $x^+ \in \partial\Omega$. The parallel transport $T_{x^+}^t$, $t \in \mathbb{R}$, in the direction of x^+ is defined by*

$$T_{x^+}^t = d\pi T^t d\pi_{x^+}^{-1}.$$

Given a vector $v \in T\Omega$, we deduce its parallel transport $T_{x^+}^t(v)$ by taking the unique vector $Z(v) \in E^s \oplus \mathbb{R}.X$ that projects on v, take its parallel transport $T^t Z(v)$ and project it again. Equivalently, since $E^s = \{Y - J^X(Y),\ Y \in VH\Omega\}$, we can also take the unique vector $h(v)$ in $\mathbb{R}.X \oplus h^X H\Omega$ that projects on v.
From Proposition 2.6, we deduce that, for any $v \in T\mathcal{H}_{x^+}$,

$$d\varphi_{x^+}^t(v) = e^{-t}T_{x^+}^t(v) \tag{3.5}$$

The only thing we have to do now is to understand the behaviour of the quantities $F(T_{x^+}^t v)$ for $v \in T\mathcal{H}_{x^+}$.

Definition 3.7. *Let $x^+ \in \partial\Omega$. The* upper and lower parallel Lyapunov exponents $\overline{\eta}(x^+, v)$ *and* $\overline{\eta}(x^+, v)$ *of a vector $v \in T\mathcal{H}_{x^+}$ in the direction of x^+, are defined as*

$$\overline{\eta}(w, v) = \limsup_{t\to+\infty} \frac{1}{t}\log F(T_{x^+}^t v),\ \underline{\eta}(w, v) = \liminf_{t\to+\infty} \frac{1}{t}\log F(T_{x^+}^t v).$$

It is now easy to see that we have the following characterization of forward regular points:

Proposition 3.8. *Let $x^+ \in \partial\Omega$ and $w = (x, [\xi]) \in W^s(x^+)$. The following are equivalent:*

- *the point w is forward regular for φ^t;*
- *the point x is forward regular for $\varphi_{x^+}^t$;*
- *there exist a splitting*

$$T_x\mathcal{H}_w = E_1 \oplus \cdots E_p \tag{3.6}$$

and numbers

$$\eta_1 < \cdots < \eta_p,$$

such that, for any $v_i \in E_i \smallsetminus \{0\}$,

$$\lim_{t\to+\infty} \frac{1}{t}\log F(T_{x^+}^t v_i) = \eta_i$$

and

$$\lim_{t\to+\infty}\frac{1}{t}\log\det T^t_{x^+} = \sum \dim E_i\ \eta_i.$$

Furthermore, the splitting (3.6) is linked by $d\pi$ to the splitting (3.4): we have $E_i = d\pi(E_i^s) = d\pi(E_i^u)$; the Lyapunov exponents χ_i^s, χ_i^u of the flow satisfy

$$\eta_i = \chi_i^s + 1 = \chi_i^u - 1.$$

The numbers η_i will be called parallel Lyapunov exponents.

4. Structure of the boundary

In this part, we give a link between parallel Lyapunov exponents and the shape of the boundary at the endpoint of the orbit. It may be a bit fastidious so you could prefer going directly to Theorem 4.9, and have a look to the part in between when it is needed.

4.1. Locally convex submanifolds of $\mathbb{RP}^n$.

Definition 4.1. *A codimension 1 $\mathcal{C}^0$ submanifold N of $\mathbb{R}^n$ is* locally (strictly) convex *if for any $x \in N$, there is a neighbourhood V_x of x in $\mathbb{R}^n$ such that $V_x \smallsetminus N$ consists of two connected components, one of them being (strictly) convex.*
A codimension 1 $\mathcal{C}^0$ submanifold N of $\mathbb{RP}^n$ is locally *(strictly)* convex *if its trace in any affine chart is locally (strictly) convex.*

Obviously, to check if $N \subset \mathbb{RP}^n$ is convex around x, it is enough to look at the trace of N in *one* affine chart at x. Choose a point $x \in N$ in a locally convex submanifold N and an affine chart centered at x. We can find a tangent space T_x of N at x such that $V_x \cap N$ is entirely contained in one of the closed half-spaces defined by T_x. We can then endow the chart with a suitable Euclidean structure, so that, around x, N appears as the graph of a convex function $f : U \subset T_x \longrightarrow [0, +\infty)$ defined on an open neighbourhood U of $0 \in T_x$. This function is (at least) as regular as N, is positive, $f(0) = 0$ and $f'(0) = 0$ if N is $\mathcal{C}^1$ at x. When N is strictly locally convex, then f is strictly convex, in particular $f(v) > 0$ for $v \neq 0$.

In what follows, we are interested in the shape of the boundary $\partial\Omega$ of Ω at some specific point, or, more generally, in the local shape of locally strictly convex $\mathcal{C}^1$ submanifolds of $\mathbb{RP}^n$. Denote by $\mathtt{Cvx}(n)$ the set of strictly convex $\mathcal{C}^1$ functions $f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $f(0) = f'(0) = 0$, where U is an open convex subset of $\mathbb{R}^n$ containing 0. We look for properties of such functions at the origin which are invariant by projective transformations.

4.2. Approximate α-regularity.

We introduce here the main notion of approximate α-regularity, describe its meaning and prove some useful lemmas.

4.2.1. *Definition.*

Definition 4.2. *A function $f \in \mathtt{Cvx}(1)$ is said to be* approximately α-regular*, $\alpha \in [1, +\infty]$, if*

$$\lim_{t\to 0}\frac{\log\dfrac{f(t)+f(-t)}{2}}{\log|t|} = \alpha.$$

This property is clearly invariant by affine transformations, and in particular by change of Euclidean structure. It is in fact invariant by projective ones, but we do not need to prove it directly, since it will be a consequence of Theorem 4.9.
Obviously, the function $t \in \mathbb{R} \mapsto |t|^\alpha$, $\alpha > 1$ is approximately α-regular. To be α-regular, with $1 < \alpha < +\infty$, means that we roughly behave like $t \mapsto |t|^\alpha$.

The case $\alpha = \infty$ is a particular one: f is ∞-regular means that for any $\alpha \geqslant 1$, $f(t) \ll |t|^\alpha$ for small $|t|$. An easy example of such a function is provided by $f : t \longmapsto e^{-1/t^2}$. On the other side, f is 1-regular means that for any $\alpha > 1$, $f(t) \gg |t|^\alpha$. An example of function which is 1-regular is provided by the Legendre transform of the last one (see section 6.2.1).

In the case where $1 < \alpha < +\infty$, we can state the following equivalent definitions. The proof is straightforward.

Lemma 4.3. *Let $f \in \texttt{Cvx}(1)$ and $1 < \alpha < +\infty$. The following propositions are equivalent:*

- *f is approximately α-regular;*
- *for any $\epsilon > 0$ and small $|t|$,*
$$|t|^{\alpha+\epsilon} \leqslant \frac{f(t)+f(-t)}{2} \leqslant |t|^{\alpha-\epsilon};$$
- *the function $t \longmapsto \dfrac{f(t)+f(-t)}{2}$ is $\mathcal{C}^{\alpha-\epsilon}$ and $\alpha+\epsilon$-convex at 0 for any $\epsilon > 0$.*

To understand the last proposition, we recall the following

Definitions 4.4. *Let $\alpha, \beta \geqslant 1$ We say that a function $f \in \texttt{Cvx}(n)$ is*

- *$\mathcal{C}^\alpha$ if for small $|t|$, there is some $C > 0$ such that*
$$f(t) \leqslant C|t|^\alpha;$$
- *β-convex if for small $|t|$, there is some $C > 0$ such that*
$$f(t) \geqslant C|t|^\beta.$$

4.2.2. *A useful equivalent definition.* We now give another equivalent definition of approximate regularity, that shows the relation with the motivation above. Theorem 4.9 is the most important consequence of it.

Let $f \in \texttt{Cvx}(1)$. Denote by $f^+ = f^{-1}_{|[0,1]}$ and $f^- = -f^{-1}_{|[-1,0]}$. These functions are both nonnegative, increasing and concave and their value at 0 is 0; they are $\mathcal{C}^1$ on $(0,1]$ and their tangent at 0 is vertical.

The harmonic mean of two numbers $a, b > 0$ is defined as
$$H(a,b) = \frac{2}{a^{-1}+b^{-1}}.$$
The harmonic mean of two functions $f,\ g : X \to (0,+\infty)$ defined on the same set X is the function $H(f,g)$ defined for $x \in X$ by
$$H(f,g)(x) = H(f(x),g(x)) = \frac{2}{\frac{1}{f(x)}+\frac{1}{g(x)}}.$$

Proposition 4.5. *A function $f \in \texttt{Cvx}(1)$ is approximately α-regular, $\alpha \in [1,+\infty]$ if and only if*
$$\lim_{t\to 0^+} \frac{\log H(f^+,f^-)(t)}{\log t} = \alpha^{-1},$$
with the convention that $\frac{1}{+\infty} = 0$.

Proof. As we will see, it is enough to take f continuous, so by replacing f^+ and f^- by $\min(f^+,f^-)$ and $\max(f^+,f^-)$, we can assume that $f^+ \leqslant f^-$, that is $f(t) \geqslant f(-t)$ for $t \geqslant 0$. Now, assuming that the limit exists,

$$\lim_{t\to 0^+} \frac{\log H(f^+,f^-)(t)}{\log t} = -\lim_{t\to 0^+} \frac{\log\left(\dfrac{1}{f^+(t)}+\dfrac{1}{f^-(t)}\right)}{\log t} = \lim_{t\to 0^+} \frac{\log f^+(t)}{\log t} - \lim_{t\to 0^+} \frac{\log\left(1+\dfrac{f^+(t)}{f^-(t)}\right)}{\log t}.$$

Since $f^+ \leqslant f^-$, the second limit is 0, and the first one is

$$\lim_{t\to 0^+} \frac{\log f^+(t)}{\log t} = \lim_{u\to 0^+} \frac{\log u}{\log f(u)}.$$

But, since $f(u) \geqslant f(-u)$ for $u \geqslant 0$, we get

$$\lim_{u\to 0^+} \frac{\log u}{\log \frac{f(u)+f(-u)}{2}} = \lim_{u\to 0^+} \frac{\log u}{\log f(u) + \log\left(1 + \frac{f(-u)}{f(u)}\right)} = \lim_{u\to 0^+} \frac{\log u}{\log f(u)},$$

hence the result. $\square$

The last construction can be generalized in a way that will be useful later, for proving Theorem 4.9. Let $f \in \mathtt{Cvx}(1)$ and pick $a > 0$. We define two new "inverse functions" $f_a^+(s)$ and $f_a^-(s)$ for $s \in [0, \epsilon]$, $\epsilon > 0$ small enough, depending on a; these are positive functions defined by the equations

$$f(f_a^+(s)) = s - sf_a^+(s); f(-f_a^-(s)) = s + sf_a^-(s).$$

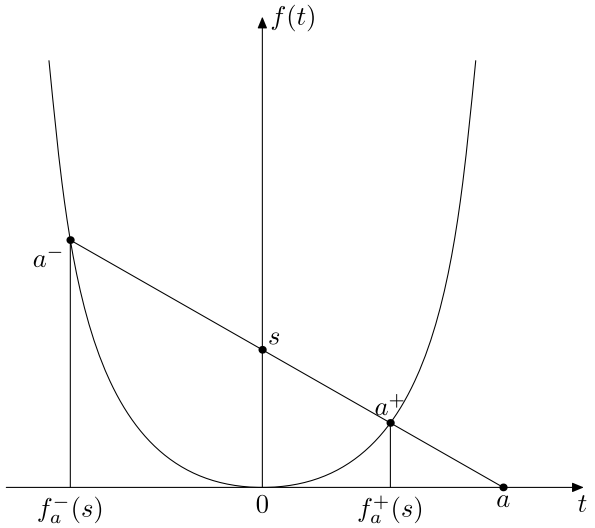

Figure 7. Construction of new inverses

Geometrically, for $s \in [0, \epsilon]$ on the vertical axis, the line (as) cuts the graph of f at two points a^+ and $a-$, with s between a^+ and a^-; $f_a^+(s)$ and $f_a^-(s)$ are the abscissae of a^+ and a^- (c.f. figure 4.2.2). f^+ and f^- can be considered as $f_{+\infty}^+$ and $f_{+\infty}^-$.

Lemma 4.6. *Let $f \in \mathtt{Cvx}(1)$ and $a > 0$. The functions $\frac{f_a^+}{f^+}$ and $\frac{f_a^-}{f^-}$ can be extended by continuity at 0 by*

$$\frac{f_a^+}{f^+}(0) = \frac{f_a^-}{f^-}(0) = 1.$$

In particular, for $s > 0$ small enough,

$$f^+(s) \asymp f_a^+(s),\ f^-(s) \asymp f_a^-(s).$$

Proof. We prove it for f^+ and f_a^+. Clearly, we have $\frac{f_a^+(s)}{f^+(s)} \leqslant 1$. Since f is convex and $f(0) = 0$, we get

$$s - sf_a^+(s) = f(f_a^+(s)) = f\left(\frac{f_a^+(s)}{f^+(s)} f^+(s)\right) \leqslant \frac{f_a^+(s)}{f^+(s)} f(f^+(s)) = \frac{f_a^+(s)}{f^+(s)} s.$$

Hence, for $0 < s \leqslant \epsilon < 1$

$$\frac{f_a^+(s)}{f^+(s)} \geqslant 1 - f_a^+(s) \geqslant 1 - f_a^+(\epsilon).$$

The function $\dfrac{f_a^+}{f^+}$ can even be extended at 0 by $\dfrac{f_a^+}{f^+}(0) = 1$ $\square$

The result to remember is the following consequence of lemmas 4.6 and 4.5:

Corollary 4.7. *Pick $a > 0$. A function $f \in \mathtt{Cvx}(1)$ is approximately α-regular if and only if*

$$\lim_{t\to 0^+} \frac{\log H(f_a^+, f_a^-)(t)}{\log t} = \alpha^{-1}.$$

4.3. **Higher dimensions.** We end this section by extending the definitions in higher dimensions:

Definitions 4.8. *A function $f \in \mathtt{Cvx}(n)$ is said to be* approximately regular *at x if it is approximately regular in any direction, that is, for any $v \in \mathbb{R}^n \setminus \{0\}$, there exists $\alpha(v) \in [1, \infty]$ such that*

$$\lim_{t\to 0} \frac{\log \dfrac{f(tv)+f(-tv)}{2}}{\log |t|} = \alpha(v).$$

Let $f \in \mathtt{Cvx}(n)$. The upper and lower Lyapunov exponents $\overline{\alpha}(v)$ and $\underline{\alpha}(v)$ of $v \in \mathbb{R}^n$ are defined by

$$\overline{\alpha}(v) = \limsup_{t\to 0} \frac{\log \dfrac{f(tv)+f(-tv)}{2}}{\log |t|},$$
$$\underline{\alpha}(v) = \liminf_{t\to 0} \frac{\log \dfrac{f(tv)+f(-tv)}{2}}{\log |t|}.$$

The function is then approximately regular if and only if the preceding limits are indeed limits in $[1, +\infty]$, that is, for any $v \in \mathbb{R}^n$, $\overline{\alpha}(v) = \underline{\alpha}(v)$. Obviously, Lemma 4.5 and Corollary 4.7 have their counterpart in higher dimensions.

4.4. **Approximate regularity of the boundary.** If Ω is a bounded convex set in the Euclidean space $\mathbb{R}^n$ with $\mathcal{C}^1$ boundary, the graph of $\partial\Omega$ at x is the function

$$\begin{array}{rccl} f: & U \subset T_x\partial\Omega & \longrightarrow & \mathbb{R}^n \\ & u & \longmapsto & \{u + \lambda n(x)\}_{\lambda\in\mathbb{R}} \cap \partial\Omega, \end{array}$$

where $n(x)$ denotes a normal vector to $\partial\Omega$ at x, and U is a sufficiently small open neighbourhood of $x \in \partial\Omega$ for the function to be defined.

We can now state our main result. Let $x^+ \in \partial\Omega$. If $w = (x, [\xi]) \in W^s(x^+)$ and $v \in T_x\mathcal{H}_w$, we denote by $p_{x^+}(v)$ the projection of v on the space $T_{x^+}\partial\Omega$ in the direction $[xx^+]$. The map p_{x^+} clearly induces an isomorphism $p_{x^+}(x)$ between each $T_x\mathcal{H}_w$ and $T_{x^+}\partial\Omega$.

Theorem 4.9. *Let Ω be a strictly convex proper open set of $\mathbb{RP}^n$ with $\mathcal{C}^1$ boundary. Pick $x^+ \in \partial\Omega$, choose any affine chart containing x^+ and a Euclidean metric on it.*
Then for any $v \in T\mathcal{H}_{x^+}$, we have

$$\overline{\eta}(x^+, v) = \frac{2}{\underline{\alpha}(x^+, p_{x^+}(v))} - 1,\ \underline{\eta}(x^+, v) = \frac{2}{\overline{\alpha}(x^+, p_{x^+}(v))} - 1,$$

where $\underline{\alpha}(x^+, p_{x^+}(v))$ and $\overline{\alpha}(x^+, p_{x^+}(v))$ denote the lower and upper Lyapunov exponents of the graph of $\partial\Omega$ at x^+ in the direction $p_{x^+}(v)$, as defined at the very end of the last section.

Proof. Let $w = (x, [\xi])$ be a point ending at x^+, $(x_t, [\xi_t]) = \varphi^t(x, [\xi])$ its image by φ^t, and $v \in T_x\mathcal{H}_w$. The vector $T_{x^+}^t v$ is at any time contained in the plane generated by ξ and v, thus, by working in restriction to this plane, we can assume that $n = 2$.
We cannot choose a good chart at w, since the chart is already fixed. But, by affine invariance, we can choose the Euclidean metric $|\ .\ |$ and ξ_t so that $\xi \perp T_{x^+}\partial\Omega = \mathbb{R}.p_{x^+}(v)$ and $|v| = |\xi_t| = 1$.

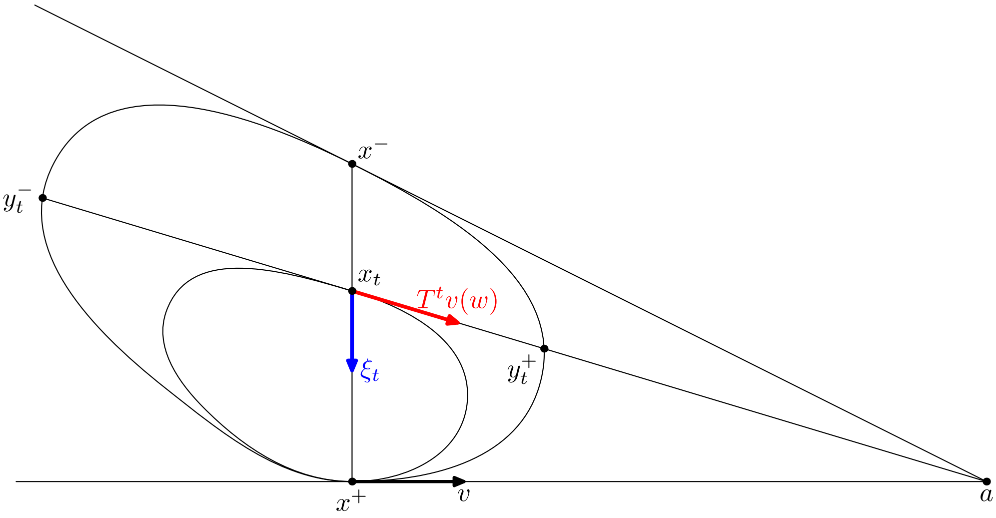

FIGURE 8. For Theorem 4.9

Let a be the point of intersection of $T_{x^+}\partial\Omega$ and $T_{x^-}\partial\Omega$. The vector $T_{x^+}v$ always points to a, that is, $T^t_{x^+}v \in \mathbb{R}.x_t a$. Thus,

$$F(T^t_{x^+}v) = \frac{|T^t_{x^+}v|}{2}\left(\frac{1}{|x_t y_t^+|} + \frac{1}{|x_t y_t^-|}\right),$$

where y_t^+ and y_t^- are the intersection points of (ax_t) and $\partial\Omega$. If $f : U \subset T_{x^+}\partial\Omega \longrightarrow \mathbb{R}$ denotes the function whose graph is a neighbourhood of x^+ in $\partial\Omega$, then

$$\frac{1}{2}\left(\frac{1}{|x_t y_t^+|} + \frac{1}{|x_t y_t^-|}\right) = \frac{1}{H(f_a^+, f_a^-)(|x_t x^+|)},$$

where f_a^+ and f_a^- are defined as in Corollary 4.7. This corollary tells us that

$$\begin{aligned}
\limsup_{t\to+\infty} \frac{1}{t}\log\frac{1}{H(f_a^+, f_a^-)(|x_t x^+|)} &= \limsup_{t\to+\infty} -\frac{\log|x_t x^+|}{t}\frac{\log H(f_a^+, f_a^-)(|x_t x^+|)}{\log|x_t x^+|}\\
&= \limsup_{t\to+\infty} -\frac{\log|x_t x^+|}{t}\limsup_{s\to 0}\frac{\log H(f_a^+, f_a^-)(s)}{\log s}\\
&= \frac{2}{\underline{\alpha}(x^+, v)}
\end{aligned}$$

(recall from Lemma 2.9 that $\lim_{t\to+\infty} \frac{1}{t}\log|x_t x^+| = -2t$). Hence

$$\limsup_{t\to+\infty} \frac{1}{t}\log F(T^t_{x^+}v) = \frac{2}{\underline{\alpha}(x^+, v)} + \limsup_{t\to+\infty}\frac{1}{t}\log|T^t_{x^+}v|.$$

From our choice of Euclidean metric, we have $|T^t_{x^+}v(w)| \asymp \langle T^t_{x^+}v(w), v\rangle$. Lemma 2.1 gives

$$T^t_{x^+}v = -L_Y m(\varphi^t w)\xi_t + (m(w)m(\varphi^t w))^{1/2} d\pi(J^{X^e}(Y)),$$

where $Y \in VH\Omega$ is such that $d\pi(J^X(Y)) = v(w)$; $d\pi(J^{X^e}(Y))$ is collinear to v and has constant Euclidean norm, which implies that

$$\lim_{t\to+\infty}\frac{1}{t}\log\langle T^t_{x^+}v, v\rangle = \lim_{t\to+\infty}\frac{1}{t}\log(m(w)m(\varphi^t w))^{1/2} = -1.$$

Hence

$$\overline{\eta}_+(w, v(w)) = \limsup_{t\to+\infty}\frac{1}{t}F(T^t_{x^+}v) = \frac{2}{\underline{\alpha}(x^+, v)} - 1.$$

Obviously, the same holds for lower exponents.

□

Theorem 4.9 tells us that the notions of approximate regularity and Lyapunov exponents are projectively invariant, that is, it makes sense for codimension 1 submanifolds of $\mathbb{RP}^n$. It then justifies the following

Definition 4.10. *A locally strictly convex $\mathcal{C}^1$ submanifold N of $\mathbb{RP}^n$ is said to be* approximately regular *at $x \in N$ if its trace in some (or, equivalently, any) affine chart at x is locally the graph of an approximately regular function. The numbers $\alpha_1(x) \geqslant \cdots \geqslant \alpha_p(x)$ attached to x are called the* Lyapunov exponents *of x.*

As a consequence, we get Theorem 1 of the introduction:

Theorem 4.11. *A point $w = (x, [\xi]) \in H\Omega$ is weakly forward regular if and only if the boundary $\partial\Omega$ is approximately regular at the endpoint $x^+ = \varphi^{+\infty}(w)$. The Lyapunov splitting of $T\mathcal{H}_{x^+}$ along $\varphi_{x^+}.x$ projects under p_{x^+} on the Lyapunov splitting of $T_{x^+}\partial\Omega$, and Lyapunov exponents are related by*

$$\eta(v) = \frac{2}{\alpha(p_{x^+}(v))} - 1, \ v \in T_x\mathcal{H}_w.$$

5. Decomposition of the horospheres and Lyapunov manifolds

From the very definition of the metric $\| \, . \, \|$ (by using remark 2.4), we get the following corollary of Theorem 4.11. Obviously, we could give an equivalent statement for non-approximately regular points by using upper and lower exponents.

Corollary 5.1. *Let Ω be a strictly convex proper open subset of $\mathbb{RP}^n$ with $\mathcal{C}^1$ boundary and fix $o \in \Omega$. Assume $x^+ \in \partial\Omega$ is approximately regular with exponents $+\infty \geqslant \alpha_1 > \cdots > \alpha_p \geqslant 1$ and filtration*

$$\{0\} = H_0 \subsetneq H_1 \subsetneq \cdots \subsetneq H_p = T_{x^+}\partial\Omega.$$

Then the horosphere $\mathcal{H}$ about x^+ passing through o admits a filtration

$$\{o\} = \mathcal{H}_0 \subsetneq \mathcal{H}_1 \subsetneq \cdots \subsetneq \mathcal{H}_p = \mathcal{H},$$

given by $\mathcal{H}_i = \{x \in \mathcal{H} \cap (H_i \oplus \mathbb{R}.ox^+)\}, \ 1 \leqslant i \leqslant p$, and such that

$$\mathcal{H}_i \smallsetminus \mathcal{H}_{i-1} = \{x \in \mathcal{H} \smallsetminus \{o\}, \ \lim_{t \to +\infty} \frac{1}{t} \log d_\Omega(\varphi^t_{x^+}(o), \varphi^t_{x^+}(x)) = \chi^s_i\},$$

with $\chi^s_i = -2 + \frac{2}{\alpha_i}$.

This allows us to define Lyapunov manifolds of the geodesic flow, that is, submanifolds tangent to the subspaces appearing in the Lyapunov filtration. In the classical theory of nonuniformly hyperbolic systems, the local existence of these manifolds is a nontrivial result traditionnally achieved with the help of Hadamard-Perron theorem. Here these manifolds appear naturally from the decomposition of the boundary at the endpoint of the orbit we are looking at. This result can be seen as a consequence of the flatness of Hilbert geometries.

Corollary 5.2. *Assume $\partial\Omega$ is approximately regular at the point x^+. Each point w of $W^s(x^+)$ is weakly forward regular with splitting*

$$TH\Omega = (\oplus_{i=1}^p E^s_i) \oplus \mathbb{R}.X \oplus (\oplus_{i=1}^p E^u_i),$$

and Lyapunov exponents

$$-2 \leqslant \chi^s_1 < \cdots < \chi^s_p \leqslant 0 \leqslant \chi^u_1 < \cdots < \chi^u_p \leqslant 2.$$

For each $w_0 = (o, [ox^+]) \in W^s(x^+)$, the stable manifold $W^s(w_0)$ admits a filtration by

$$\{w_0\} \subsetneq W_1^s(w_0) \subsetneq \cdots \subsetneq W_p^s(w_0) = W^s(w_0),$$

with

$$\begin{aligned} W_i^s(w) &:= \{w = (x, [xx^+]) \in W^s(w_0),\ x \in \mathcal{H}_i\} \\ &= \{w \in H\Omega,\ \limsup_{t\to+\infty} \frac{1}{t} \log d_{H\Omega}(\varphi^t(w_0), \varphi^t(w)) \leqslant \chi_i^-\}. \end{aligned}$$

The tangent distribution to $W_i^s(w)$ is precisely $\oplus_{k=1}^i E_k^s$.

Obviously, the last corollary can be stated also for an approximately regular point $x^- \in \partial\Omega$ and the corresponding unstable manifold

$$W^u(x^-) = \{w \in H\Omega,\ \varphi^{-\infty}(w) = x^-\}.$$

5.1. **Non-strict convexity, non-$\mathcal{C}^1$ points.** We now explain how to extend Corollary 5.1 to an arbitrary convex set. Let Ω be *any* convex proper open subset of $\mathbb{RP}^n$ and choose a point $x^+ \in \partial\Omega$. The flow $\varphi_{x^+}^t$ is well defined, the definition of approximate regularity given in section 4 still makes sense and the results we achieve before can be extended to this general convex set by using the following easy lemma.

Lemma 5.3. *Let Ω be any proper convex subset of $\mathbb{RP}^n$ and $x \in \partial\Omega$.*

- *The maximal flat*
$$\mathcal{F}(x) = \{y \in \partial\Omega,\ [xy] \subset \partial\Omega\}$$
containing x in $\partial\Omega$ is a closed convex subset of a projective subspace $\mathbb{RP}^q$, for some $0 \leqslant q \leqslant n-1$, whose interior is open in this $\mathbb{RP}^q$ when $\mathcal{F}(x)$ is not reduced to $\{x\}$.
- *The set of C^1 directions*
$$\mathcal{D}(x) = \{0\} \cup \{v \in T_x\partial\Omega \smallsetminus \{0\},\ \partial\Omega \text{ is differentiable in the direction } v\}$$
is a subspace of $T_x\partial\Omega$.

Proof. The set $\mathcal{F}(x)$ is obviously closed. It is convex because of the convexity of Ω. The projective subspace $\mathbb{RP}^q$ is the one spanned by $\mathcal{F}(x)$. The second point is just a consequence of convexity. □

Choose a direction $v \in T_x\partial\Omega$ in which the boundary $\partial\Omega$ is not differentiable and any vector $u \notin T_x\partial\Omega$. We can consider the 2-dimensional convex set $C_v(w) = \Omega \cap (\mathbb{R}.v \oplus \mathbb{R}.u)$. It is easy to see that, for two distinct geodesic lines of $C_v(u)$ ending at x, the distance between them does not tend to 0. Hence the negative Lyapunov exponent χ^s of such a geodesic, if it were defined, would be $\chi^s = 0$; it is coherent with the fact that $\alpha(v) = 1$ and the relation $\chi^s = -2 + \frac{2}{\alpha(v)}$.
We can now consider the subspace $\mathcal{D}(x)$ of $\mathcal{C}^1$ directions and the convex set $C_x(u) = \Omega \cap (D(x) \oplus \mathbb{R}.u)$ for an arbitrary vector $u \notin T_x\partial\Omega$. For example, the stable manifold $\mathcal{H}_{x^+}^s(x)$ of $\varphi_{x^+}^t$ at x is the set

$$\mathcal{H}_{x^+}^s(x) = C_x(xx^+) \cap \mathcal{H}_{x^+}(x).$$

The boundary $\partial C_x(u)$ is $\mathcal{C}^1$ at Lebesgue-almost every point x^-, so all we did before is relevant along Lebesgue almost-all geodesic (x^-x^+). We just have to be careful for those vectors in span $\mathcal{F}(x)$ which were not considered before: in such a direction v, the boundary is obviously $+\infty$-approximately regular, and the distance between two geodesics of $C_v(u)$ with origin on the same horosphere and ending at x^+ goes to 0 as e^{-2t}.

As a consequence, we get that Corollary 5.1 is valid for any Hilbert geometry:

Corollary 5.4. *Let Ω be a convex proper open subset of $\mathbb{RP}^n$ and fix $o \in \Omega$. Assume $x^+ \in \partial\Omega$ is approximately regular with exponents $+\infty \geqslant \alpha_1 > \cdots > \alpha_p \geqslant 1$ and filtration*

$$\{0\} = H_0 \subsetneq H_1 \subsetneq \cdots \subsetneq H_p = T_{x^+}\partial\Omega.$$

Then the horosphere $\mathcal{H} = \mathcal{H}_{x^+}(o)$ about x^+ passing through o admits a filtration

$$\{o\} = \mathcal{H}_0 \subsetneq \mathcal{H}_1 \subsetneq \cdots \subsetneq \mathcal{H}_p = \mathcal{H},$$

given by $\mathcal{H}_i = \{x \in \mathcal{H} \cap (H_i \oplus \mathbb{R}.ox^+)\}$, $1 \leqslant i \leqslant p$, such that

$$\mathcal{H}_i \smallsetminus \mathcal{H}_{i-1} = \{x \in \mathcal{H} \smallsetminus \{o\},\ \lim_{t\to+\infty} \frac{1}{t} \log d_\Omega(\varphi^t_{x^+}(o), \varphi^t_{x^+}(x)) = \chi^s_i\},$$

with $\chi^s_i = -2 + \frac{2}{\alpha_i}$.

In this last corollary, if $\mathcal{F}(x^+)$ is not reduced to x^+, then the subspace H_1 itself admits a filtration $\{0\} \subsetneq \text{span } \mathcal{F}(x^+) \subsetneq H_1$; $H_1 \smallsetminus \text{span } \mathcal{F}(x^+)$ consists of these vectors v with Lyapunov exponent $\alpha(v) = +\infty$ which are not in span $\mathcal{F}(x^+)$, that is, the directions in which $\partial\Omega$ is not flat, but infinitesimally flat. Of course this also provides a filtration of $\mathcal{H}_1$.
Similarly, if $\mathcal{D}(x^+)$ is not all of $T_{x^+}\partial\Omega$, we can refine the filtration into

$$\cdots \subsetneq H_{p-1} \subsetneq \mathcal{D}(x^+) \subsetneq H_p = T_{x^+}\partial\Omega.$$

The subspace $\mathcal{D}(x^+)$ is precisely the tangent space to the stable manifold $\mathcal{H}^s_{x^+}(o)$ of $\varphi^t_{x^+}$ at o, and $\mathcal{H}$ admits a subfiltration

$$\cdots \subsetneq \mathcal{H}_{p-1} \subsetneq \mathcal{H}^s_{x^+}(o) \subsetneq \mathcal{H}_p = \mathcal{H}.$$

6. More about approximate regularity

Despite its naturality, the notion of approximate regularity seems to be new. I do not know what can be said in general about it. I give some first results in the next two sections, and raise questions about it in a third one.

6.1. **The decomposition theorem.**

We first remark that Theorem 4.9 allows us to understand more about approximately regular functions, yielding a somehow surprising decomposition result:

Theorem 6.1. *Let $f \in \texttt{Cvx}(n)$. Then*

- *the numbers $\overline{\alpha}(v)$, $v \in \mathbb{R}^n \smallsetminus \{0\}$, can take only a finite numbers of values. More precisely, there exist a number $\overline{p}$, a filtration*

$$\{0\} = \overline{H}_0 \subsetneq \overline{H}_1 \subsetneq \cdots \subsetneq \overline{H}_{\overline{p}} = \mathbb{R}^n$$

  and numbers

$$+\infty \geqslant \overline{\alpha}_1 > \cdots > \overline{\alpha}_{\overline{p}} \geqslant 1,$$

  such that for any $v_i \in \overline{H}_i \smallsetminus \overline{H}_{i-1}$, $1 \leqslant i \leqslant \overline{p}$,

$$\limsup_{t\to 0} \frac{\log \dfrac{f(tv_i) + f(-tv_i)}{2}}{\log |t|} = \overline{\alpha}_i.$$

  The same holds for lower Lyapunov exponents.

- *the following propositions are equivalent:*
  (i) *f is approximately regular;*
  (ii) *there exist a splitting $\mathbb{R}^n = \oplus_{i=1}^p G_i$ and numbers $+\infty \geqslant \alpha_1 > \cdots > \alpha_p \geqslant 1$ such that the restriction $f|_{G_i \cap U}$ is approximately regular with exponent α_i;*
  (iii) *there exist a filtration*

$$\{0\} = H_0 \subsetneq H_1 \subsetneq \cdots \subsetneq H_p = \mathbb{R}^n$$

  and numbers $+\infty \geqslant \alpha_1 > \cdots > \alpha_p \geqslant 1$ such that, for any $v_i \in H_i \smallsetminus H_{i-1}$, the restriction $f|_{\mathbb{R}.v_i \cap U}$ is approximately regular with exponent α_i.

When f is approximately regular, we call the numbers α_i the Lyapunov exponents of f.

Proof. The graph of f can always be considered as the boundary of a strictly convex set $\Omega \subset \mathbb{R}^{n+1}$ with $\mathcal{C}^1$ boundary. We can then apply Theorem 4.9 to this set Ω. $\square$

6.2. **Duality and approximate regularity.**

6.2.1. *Legendre transform.* Pick a function $f \in \mathtt{Cvx}(n)$. Since f is $\mathcal{C}^1$ and strictly convex, the gradient

$$\nabla : x \in U \longmapsto \nabla_x f = \left(\frac{\partial f}{\partial x_1}(x), \cdots, \frac{\partial f}{\partial x_n}(x)\right)$$

is an injective map onto a convex subset V of $\mathbb{R}^n$. Using the gradient, a point x can thus be defined by its coordinates $(x_1, \cdots, x_n)$ or by its "dual" coordinates $\left(\frac{\partial f}{\partial x_1}(x), \cdots, \frac{\partial f}{\partial x_n}(x)\right)$.
The **Legendre transform** of f is the function f^* defined by

$$f(x) + f^*(\nabla_x f) = \langle \nabla_x f, x \rangle.$$

It happens that the transform $f \longmapsto f^*$ is an involution of $\mathtt{Cvx}(n)$. We will see in the next section that it appears naturally when one considers the dual of a convex set. Our goal in the next section is to make a link between the shape of the boundary of the convex set and the one of its dual. For this, we study here the link between the approximate regularity of f and of its Legendre transform f^*. I am not very familiar with Legendre transform and I did not manage to prove the next lemma in higher dimensions; but it is probably true...

Lemma 6.2. *Assume $f \in \mathtt{Cvx}(1)$ is approximately α-regular, $\alpha \in [1, +\infty]$. Then the Legendre transform f^* of f is approximately α^*-regular with*

$$\frac{1}{\alpha^*} + \frac{1}{\alpha} = 1.$$

Proof. We only prove the proposition when $\alpha \in (1, +\infty)$. The Legendre transform of $f \in \mathtt{Cvx}(1)$ is given by

$$f^*(f'(x)) = xf'(x) - f(x).$$

By considering $f(x) + f(-x)$ instead, we can assume that f is an even function, so that approximate α-regularity gives

$$\lim_{x \to 0^+} \frac{\log f(x)}{\log x} = \alpha.$$

Since $f(0) = f(x) - xf'(x) + o(x)$, we get

$$\lim_{x \to 0^+} \frac{\log f'(x)}{\log x} = \alpha - 1.$$

We need to understand the limit

$$\lim_{x \to 0^+} \frac{\log xf'(x) - f(x)}{\log f'(x)}.$$

Fix $\epsilon > 0$. There is some $x > 0$ such that for $0 \leqslant t \leqslant x$, we have

$$t^{\alpha-1+\epsilon} \leqslant f'(t) \leqslant t^{\alpha-1-\epsilon}. \tag{6.1}$$

Remark that

$$xf'(x) - f(x) = xf'(x) - \int_0^x f'(t)dt.$$

From (6.1), that means the value of $xf'(x) - f(x)$ is in between the two areas between 0 and x delimited by the line $y = f'(x)$ above and, respectively, the curves $t \mapsto t^{\alpha-1-\epsilon}$ and $t \mapsto t^{\alpha-1+\epsilon}$ below:

$$f'(x)^{\frac{1}{\alpha-1-\epsilon}} f'(x) - \int_0^{f'(x)^{\frac{1}{\alpha-1-\epsilon}}} t^{\alpha-1-\epsilon}\, dt \leqslant xf'(x) - f(x) \leqslant xf'(x) - \int_0^x t^{\alpha-1+\epsilon}\, dt.$$

Hence
$$(f'(x))^{\frac{\alpha-\epsilon}{\alpha-1-\epsilon}} - \frac{1}{\alpha-\epsilon}(f'(x))^{\frac{\alpha-\epsilon}{\alpha-1-\epsilon}} \leqslant xf'(x) - f(x) \leqslant xf'(x) - \frac{1}{\alpha+\epsilon}x^{\alpha+\epsilon}.$$
Using (6.1) again, we get
$$\frac{\alpha-\epsilon-1}{\alpha-\epsilon}(f'(x))^{\frac{\alpha-\epsilon}{\alpha-1-\epsilon}} \leqslant xf'(x) - f(x) \leqslant x^{\alpha-\epsilon} - \frac{1}{\alpha+\epsilon}x^{\alpha+\epsilon} \leqslant x^{\alpha-\epsilon}.$$
So
$$\frac{\alpha-\epsilon}{\alpha-1} = (\alpha-\epsilon)\lim_{x\to 0^+}\frac{\log x}{\log f'(x)} \leqslant \lim_{x\to 0^+}\frac{\log xf'(x)-f(x)}{\log f'(x)} \leqslant \frac{\alpha-\epsilon}{\alpha-1-\epsilon}.$$
Since ϵ is arbitrary small, we get the result. □

6.2.2. *Dual convex set.* To each convex set $\Omega \subset \mathbb{RP}^n$ is associated its dual convex set Ω^*. To define it, consider one of the two convex cones $C \subset \mathbb{R}^{n+1}$ whose trace is Ω. The dual convex set Ω^* is the trace of the dual cone
$$C^* = \{f \in (\mathbb{R}^{n+1})^*,\ \forall x \in C,\ f(x) > 0\}.$$
The cone C^* is a subset of the dual of $\mathbb{R}^{n+1}$ but of course, it can be seen as the subset
$$\{y \in \mathbb{R}^{n+1},\ \forall x \in C,\ \langle x, y\rangle \geqslant 0\}.$$
The set Ω^* can be identified with the set of projective hyperplanes which do not intersect $\overline{\Omega}$: to such a hyperplane corresponds the line of linear maps whose kernel is the given hyperplane. For example, we can see the boundary of Ω^* as the set of tangent spaces to $\partial\Omega$. In particular, when Ω is strictly convex with $\mathcal{C}^1$ boundary, there is a homeomorphism between the boundaries of Ω and Ω^*: to the point $x \in \partial\Omega$ we associate the (projective class of the) linear map x^* such that $\ker x^* = T_x\partial\Omega$.

In the following we would like to link the shape of $\partial\Omega$ and $\partial\Omega^*$. We will work in $\mathbb{R}^{n+1}$ with the cones C and C^* where it is more usual to make computations. Choose a point $p \in \partial C$ and fix a Euclidean structure on $\mathbb{R}^{n+1}$ and an orthonormal basis $(u_1, \cdots, u_{n+1})$ so that $p = u_1 + u_{n+1}$, $T_p\partial C = \mathrm{span}\{p, u_2, \cdots, u_{n+1}\}$ and $C \subset \{x = (x_1, \cdots, x_{n+1}),\ x_{n+1} > 0\}$. We identify Ω with the intersection $C \cap \{x_{n+1} = 1\}$ and the tangent space $T_p\partial\Omega$ is $p + \mathrm{span}\{u_2, \cdots, u_{n+1}\}$.
Call $f : U \subset T_p\partial\Omega \longrightarrow \mathbb{R}$ the local graph of $\partial\Omega$ at p, such that, around p,
$$\partial\Omega = \{(1 - f(x_2, \cdots, x_n), x_2, \cdots, x_n, 1)\}.$$

Lemma 6.3. *Around $p^* = (1, 0, \cdots, 0, -1)$, the boundary $\partial\Omega^*$ is given by*
$$\partial\Omega^* = \{(1, \lambda_2, \cdots, \lambda_n, -1 - f^*(\lambda_2, \cdots, \lambda_n))\},$$
where f^* is the Legendre transform of f. In other words, the local graph of $\partial\Omega^*$ at p is given by the Legendre tranform f^* of f.

Proof. Take a point $x = (1-f(x_2, \cdots, x_n), x_2, \cdots, x_n, 1) \in \partial\Omega$ and call $x_{2n} = x_2u_2+\cdots+x_nu_n \in \mathrm{span}\{u_2, \cdots, u_{n+1}\}$ its projection on $\mathrm{span}\{u_2, \cdots, u_{n+1}\}$. Call $F : T_p\partial\Omega \longrightarrow \mathbb{R}^{n+1}$ the map given by
$$F(p + x_{2n}) = p + x_{2n} - f(x_{2n})u_1 = (1 - f(x_2, \cdots, x_n), x_2, \cdots, x_n, 1).$$
The tangent space of $\partial\Omega$ at x is then given by
$$T_x\partial\Omega = x + d_xF(\mathrm{span}\{u_2, \cdots, u_{n+1}\}).$$
But, for $h \in \mathrm{span}\{u_2, \cdots, u_{n+1}\}$, we have $d_xF(h) = -d_xf(h) + h$. Hence
$$T_x\partial\Omega = x + \{h - d_xf(h),\ h \in \mathrm{span}\{u_2, \cdots, u_{n+1}\}\}.$$
Now the dual point of x is the linear map $x^* = (x_1^*, \cdots, x_{n+1}^*)$ such that $x^*(x) = 0$, $x^*(T_x\partial\Omega) = 0$ and $x^*(u_1) = 1$. (This last condition is just a normalization condition, since there is a line of corresponding linear maps.) The third condition gives $x_1^* = 1$. The second implies that for any $h \in \mathrm{span}\{u_2, \cdots, u_{n+1}\}$,
$$0 = x^*(h - d_xf(h)) = \langle x^* - \nabla_xf, h\rangle;$$

hence $x^*_{2n} = \nabla_x f$, that is, $x^*_i = \frac{\partial f}{\partial u_i}(x_{2n}),\ i = 2, \cdots, n$. Finally, the first condition gives

$$1 - f(x_2, \cdots, x_n) + \langle \nabla_x f, x_{2n} \rangle + x^*_{n+1} = 0,$$

so

$$x^*_{n+1} = -1 - (\langle \nabla_x f, x_{2n} \rangle - f(x_{2n})).$$

By considering the set of variables $(\lambda_2, \cdots, \lambda_n) = \left(\frac{\partial f}{\partial u_2}, \cdots, \frac{\partial f}{\partial u_n}\right)$, one finally gets

$$x^* = (1, \lambda_2, \cdots, \lambda_n, -1 - f^*(\lambda_2, \cdots, \lambda_n)).$$

□

From Lemma 6.2, we get the following

Corollary 6.4. *Assume $\partial\Omega \subset \mathbb{RP}^2$ is approximately α-regular at the point x. Then $\partial\Omega^*$ is approximately α^*-regular at the point x^* with*

$$\frac{1}{\alpha^*} + \frac{1}{\alpha} = 1.$$

6.3. **Questions.** The first thing to say is that one can construct "by hand" a function $f \in \mathtt{Cvx}(1)$ which is not approximately regular. In a forthcoming paper [Cra], I will also show that the boundary of a divisible strictly convex Hilbert geometry is not approximately regular at some points. (Recall that a Hilbert geometry is divisible if it admits a discrete cocompact subgroup of isometries.)
Nevertheless, one could expect this situation to be quite exceptional. For example, in [Cra], we will also see that the boundary of a divisible strictly convex Hilbert geometry is Lebesgue almost everywhere approximately regular with the same exponents. This raises the following

Question. *Is any convex function $f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ approximately regular at Lebesgue almost every point of U?*

A theorem of Alexandrov [Ale39] asserts that a convex function $f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ is twice differentiable at Lebesgue almost every point. This implies that for Lebesgue almost every $x \in U$ and every $v \in \mathbb{R}^n$, we have $\underline{\alpha}(x, v) \geqslant 2$. But it does not imply much more.

In the 1-dimensional case, that is for a convex function $f : U \subset \mathbb{R} \longrightarrow \mathbb{R}$, one has only one lower and one upper Lyapunov exponent at each point $x \in U$. It could be interesting to understand, for $\alpha, \beta \in [1, +\infty]$ the sets

$$\underline{K_\alpha}(f) = \{x \in I,\ \underline{\alpha}(x) \geqslant \alpha\} \text{ and } \overline{K_\beta}(f) = \{x \in I,\ \overline{\alpha}(x) \leqslant \beta\},$$

as well as their intersections; one could look at their Lebesgue measure, their Hausdorff dimension (or other dimensions)...
Equivalently, in higher dimensions, we could look, for open subset V, V' of $\mathbb{R}^n$, at the sets

$$\underline{K_V}(f) = \{x \in I,\ \underline{\alpha}(x) \in V\} \text{ and } \overline{K_{V'}}(f) = \{x \in I,\ \overline{\alpha}(x) \in V'\},$$

where $\underline{\alpha}(x)$ and $\overline{\alpha}(x)$ denote the vector of lower and upper Lyapunov exponents of f at x, counted with multiplicities.

The special case of the boundary of a divisible strictly convex Hilbert geometry, that will be studied in [Cra], may give an idea of the range of possibilities.

7. About volume entropy

The volume entropy of a Riemannian metric g on a manifold M measures the asymptotic exponential growth of the volume of balls in the universal cover $\tilde{M}$; it is defined by

$$h_{vol}(g) = \limsup_{R\to+\infty} \frac{1}{r} \log vol_g(B(x,R)), \tag{7.1}$$

where vol_g denotes the Riemannian volume corresponding to g. We define the volume entropy of a Hilbert geometry (Ω, d_Ω) by the same formula, with respect the Busemann volume.

Some results are already known: for instance, if Ω is a polytope then $h_{vol}(\Omega, d_\Omega) = 0$; at the opposite, we have the

Theorem 7.1 ([BBV])**.** *Let $\Omega \subset \mathbb{RP}^n$ be a convex proper open set. If the boundary $\partial\Omega$ of Ω is $\mathcal{C}^{1,1}$, that is, has Lipschitz derivative, then $h_{vol}(\Omega, d_\Omega) = n-1$.*

The global feeling is that any Hilbert geometry is in between the two extremal cases of the ellipsoid and the simplex. In particular, the following conjecture is still open:

Conjecture 7.2. *For any $\Omega \subset \mathbb{RP}^n$,*

$$h_{vol}(\Omega, d_\Omega) \leqslant n-1.$$

In [BBV] the conjecture is proved in dimension $n = 2$ and an example is explicitly constructed where $0 < h_{vol} < 1$. Following their idea for proving Theorem 7.1, we can get the

Proposition 7.3. *Let (Ω, d_Ω) be any Hilbert geometry, and $\mathcal{L}$ a probability Lebesgue measure on $\partial\Omega$. Then*

$$h_{vol} \geqslant \int \frac{2}{\underline{\alpha}}\ d\mathcal{L},$$

where $\underline{\alpha}$ is defined by

$$\frac{1}{\underline{\alpha}(x)} = \sum_{i=1}^{n} \frac{1}{\underline{\alpha_i}(x)},\ x \in \partial\Omega,$$

with $\underline{\alpha_1}(x) \geqslant \cdots \geqslant \underline{\alpha_n}(x)$ being the lower¡ Lyapunov exponents at x, counted with multiplicity.

Proof. In [BBV], the authors proved that h_{vol} also measures the exponential growth rate of the volume of spheres:

$$h_{vol} = \limsup_{R\to+\infty} \frac{1}{R} \log vol(S(o,R)),$$

where $S(o,R) = \{x \in \Omega,\ d_\Omega(o,x) = R\}$ is the sphere of radius R about the arbitrary point o, and vol denotes the Busemann volume on the sphere. This is well defined because metric balls are convex, hence $S(o,R)$ is $\mathcal{C}^1$ Lebesgue-almost everywhere, so we can consider the Finsler metric induced by F on $S(o,R)$ and define Busemann volume.
Fix a probability Lebesgue measure $d\xi$ on the set of directions $H_o\Omega$ about the point o, that we identify with the unit sphere $S(o,1)$. The volume of the sphere $S(o,R)$ is then given by

$$vol(S(o,R)) = \int f(\xi,R)\ d\xi,$$

where $f(\xi,R) = \det dF(\xi,R)$ with F being the projection about o from $S(o,1)$ to $S(o,R)$. Now, using Jensen inequality and the concavity of log, we get that

$$h_{vol} \geqslant \limsup_{R\to+\infty} \int \frac{1}{R} \log f(\xi,R)\ d\xi;$$

then, the dominated convergence theorem gives

$$h_{vol} \geqslant \int \limsup_{R\to+\infty} \frac{1}{R} \log f(\xi,R)\ d\xi.$$

But it is not difficult to see that, almost everywhere,

$$\limsup_{R\to+\infty} \frac{1}{R} \log f(\xi, R) = \overline{\chi}^+(o,\xi) = \frac{2}{\underline{\alpha}(\xi^+)},$$

with $\xi^+ = \varphi^{+\infty}(o,[\xi])$. Hence the result. $\square$

As a corollary, we can for example state the following result:

Corollary 7.4. *Let (Ω, d_Ω) be any Hilbert geometry. If the boundary $\partial\Omega$ is β-convex for some $1 \leqslant \beta < +\infty$ then $h_{vol} > 0$.*

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