AN UNDERSTANDABLE APPROACH TO MY THESIS

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1. Perceptions

Let me present my personal approach on my work.

The general problem is the one of perception. In the real life, we see objects, we can touch them, they have a shape, but do they exist outside of our own perception ? do other people see them as I do ?

Mathematicians see two main issues in these questions:

- define *what* is an object, by an abstract and consistent definition;
- study representations of the same abstractly defined object, the different ways one can look at it.

What is funny is that most abstract definitions also rely on some perception we have of the object. The objects that I study in my work are **manifolds**, which are defined, roughly, by saying that locally, they *look like* the n-dimensional real space $\mathbb{R}^n$, which can be algebraically constructed.
Remark that there is often a difficult question once one has given a definition: *construct* an object that satisfies the definition. And one way of doing that is to find a geometric object that is a specific representation of the abstract one.
For example, one can define what is a sphere as an abstract manifold. Of course, one geometric representation of this sphere is the one we have in mind, the round sphere, the surface of a ball. This is a representation in our *real* 3*-dimensional world* $\mathbb{R}^3$. We can deform this representation, just by deforming the shape of the round sphere. This deformed object is the object as manifold but its geometric representation is different. My work is related to these geometric representations.

2. Hyperbolic and Hilbert geometries

Given an abstract smooth manifold M, we want to compare the different *geometric representations* of M. Here one has to precise what one means by *geometric representation*: this is a way of restricting the problem. In my work, we begin with one specific representation of the given manifold M, as a *hyperbolic* manifold. That means that our manifold locally looks like the hyperbolic space; equivalently, it carries a Riemannian metric of constant negative curvature -1.

The Beltrami model of the hyperbolic space $\mathbb{H}^n$ is the metric space (Ω_0, d). Blablabla. The hyperbolic space is homogeneous: its group of automorphisms is transitive. In other words, it looks the same at every point. Thus, a hyperbolic manifold is also very homogeneous: we cannot say where we are on the manifold just by looking around us, because in a hyperbolic manifold, the world around us is everywhere the same.

To have an example in mind, one should think of a surface of higher genus $g \geqslant 2$, that can carry a hyperbolic structure. Now, as we did for the sphere, we are going to deform this geometric structure. One way could be to make the curvature vary (that is what was done with the sphere), but we could also leave the Riemannian world to enter a new one. In my work, we see hyperbolic geometry as one particular Hilbert geometry. A Hilbert geometry is a metric space (Ω, d_Ω) where. This is not a Riemannian space but a Finsler one. The metric is not defined by a quadratic form on each tangent space but by a norm: When the convex set Ω is an ellipsoid, we recover the Beltrami model of the hyperbolic space.

3. The geodesic flow

The question is now to understand the differences between the geometric representations of the same manifold M in hyperbolic geometry or in Hilbert geometry. I restricted my study to those Hilbert geometries which are defined by strictly convex sets with C^1 boundary because they are those which have some hyperbolic behaviour, and also because they are the easiest to deal with. My work deals with one particular way of comparing the geometries: I look at how one can *move* on the geometric manifold, by studying their geodesic flow.
Geodesics are those paths in (Ω, d_Ω) which are the shortest paths between any two points. It happens that they are exactly the lines, followed at unit (Finsler) speed. Then, given a pair $w = (x, [\xi])$ consisting of a point $x \in \Omega$ and a direction $[\xi]$, there is only one geodesic leaving x in the direction $[\xi]$. So we can define the geodesic flow

$$\varphi^t : HM \longrightarrow HM$$

for any manifold $M = \Omega/\Gamma$. The goal is to compare the properties of φ^t when the geometry is hyperbolic and when it is not.

There are local and global questions. One can look at a specific orbit and how the flow behaves around it, or one can look at all points at the same time and describe their global behaviour.

4. Local properties

We here take one specific point $w = (x, [\xi]) \in H\Omega$, and we want to study what happens asymptotically around the orbit of this point. For this, we look at how the flows expands or contracts distances around this orbit by looking at the quantities $\|d\varphi^t Z\|$ when t is large and Z is vector tangent to $H\Omega$ at the point w. The norm $\|\cdot\|$ is a Finsler norm on $H\Omega$ naturally defined via the Finsler metric F on Ω; in this way, distances on $H\Omega$ are closely related to distances on Ω.

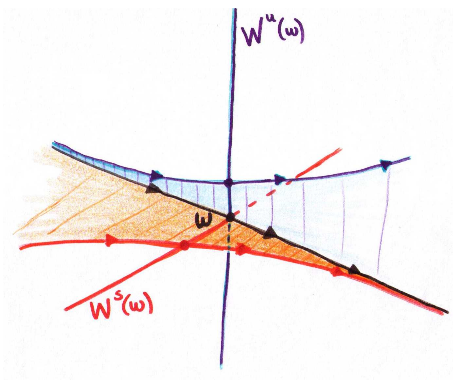


Figure 1. Local behaviour

The first observation is the following: the tangent bundle $TH\Omega$ admits a decomposition into

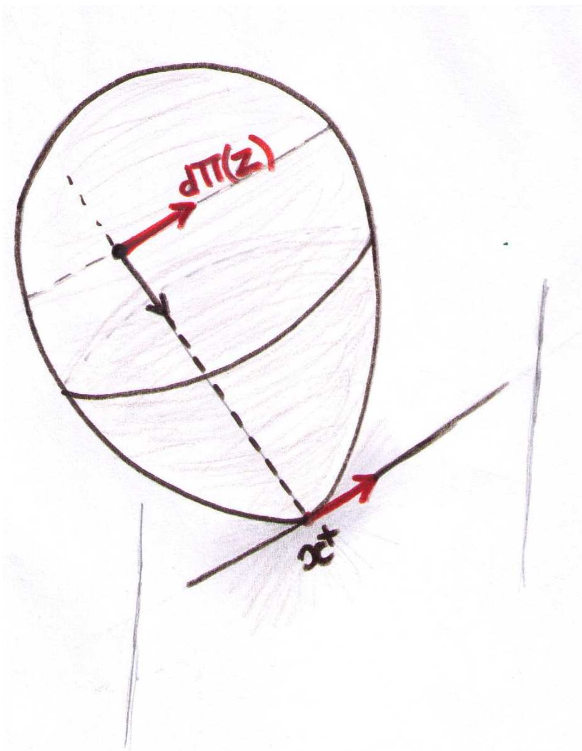
$$TH\Omega = \mathbb{R}.X \oplus E^s \oplus E^u$$

which is invariant under the flow and such that

- X is the vector field that generates the geodesic flow;

- $t \longmapsto \|d\varphi^t Z^s\|$ is strictly decreasing from $\mathbb{R}$ to $(0, +\infty)$, $Z^s \in E^s$;
- $t \longmapsto \|d\varphi^t Z^u\|$ is strictly increasing from $\mathbb{R}$ to $(0, +\infty)$, $Z^u \in E^u$.

The picture is then the following: around the orbit of the point w, there are two other "directions", E^s and E^u, along which the flow contracts or expands distances. In other words, there are some orbits that come closer and closer to the orbit of w, and other that go away (see figure 4). For this reason, the distribution E^s is called the stable distribution and E^u the unstable one.
We then would like to be more precise and to understand how distances are indeed contracted or expanded by looking at the asymptotic behaviour of $\|d\varphi^t Z\|$ when t is large, and Z is in E^s or E^u. A first step is to understand when distances are exponentially contracted (or expanded).



Actually, this property, as well as the exponential rates of contraction/expansion, can be read on the shape of the boundary $\partial\Omega$ at the point x^+ that the orbit finally reaches. We can make a precise statement of this but let me instead give an easy example.
First of all, the flatness of the geometry (essentially the fact that geodesics are lines) implies that when one picks a tangent vector Z at w, the image $d\varphi^t(Z)$ of Z by the flow is always in the 2-plane containing x, ξ and the projection $d\pi(Z)$ of Z on $T\Omega$. So, what is important is the convex set defined by the intersection of Ω with this plane, and we can thus restrict to the 2-dimensional case (see figure 4).
Now, an easy example is given by the following: if for some $1 \leqslant \alpha \leqslant +\infty$ and any $\varepsilon > 0$, $\partial\Omega$ is $C^{\alpha-\epsilon}$ and $\alpha + \epsilon$-convex at x^+, then

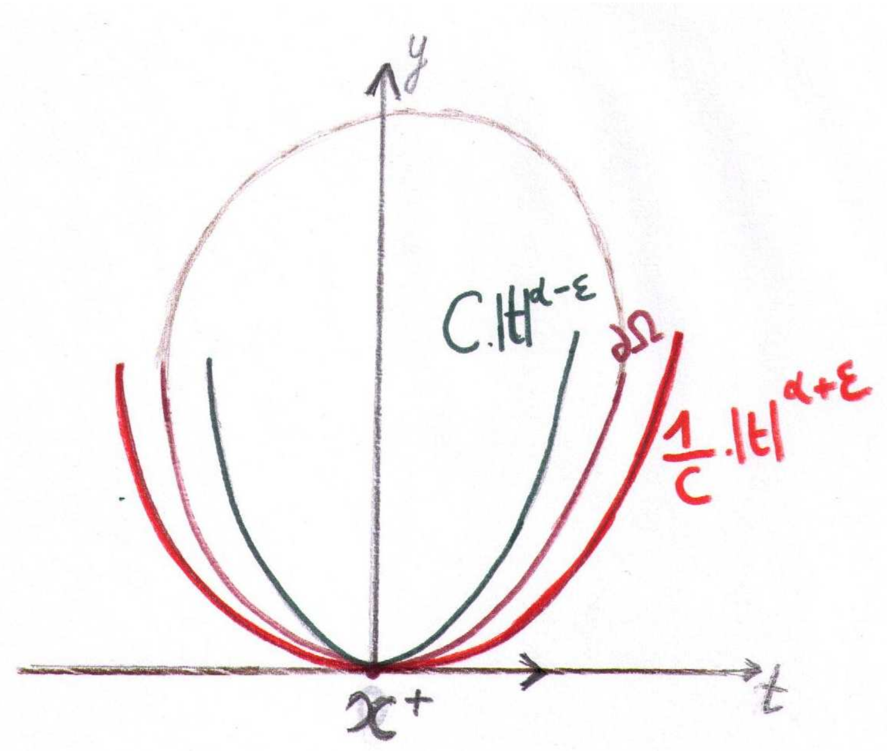
$$\lim_{t\to+\infty} \frac{1}{t} \log \|d\varphi^t Z^s\| = -2 + \frac{2}{\alpha} \quad \text{for } Z^s \in E^s,$$

$$\lim_{t\to+\infty} \frac{1}{t} \log \|d\varphi^t Z^u\| = \frac{2}{\alpha} \quad \text{for } Z^u \in E^u.$$

The assumption that $\partial\Omega$ is $C^{\alpha-\epsilon}$ and $\alpha+\epsilon$-convex at x^+ for any $\epsilon>0$ means that there are constant C_ϵ such that the graph f of $\partial\Omega$ at x^+ satisfies

$$\frac{1}{C}|t|^{\alpha+\epsilon} \leqslant f(t) \leqslant C|t|^{\alpha-\epsilon}$$

for small t:



5. Global behaviour

5.1. **Compact manifolds.** The understanding of the local behaviour allows to prove the following

Theorem (Yves Benoist)**.** *Let $M=\Omega/\Gamma$ be compact. Then the geodesic flow is an Anosov flow with decomposition*

$$THM = \mathbb{R}.X \oplus E^s \oplus E^u;$$

that is, there exist constants $C,\alpha,\beta>0$ such that for any $t\geqslant 0$,

$$\|d\varphi^t(Z^s)\| \leqslant Ce^{-\alpha t}\|Z^s\|,\ Z^s\in E^s,$$
$$\|d\varphi^t(Z^u)\| \leqslant Ce^{-\beta t}\|Z^u\|,\ Z^u\in E^u$$

The Anosov paroperty is an important property of the geodesic flow of a hyperbolic metric. But our geodesic flow does not share all the properties of the geodesic flow of a hyperbolic metric.

Theorem 1. *Let $M=\Omega/\Gamma$ be compact. Then the topological entropy h_{top} of the geodesic flow on HM satisfies*

$$h_{top} \leqslant n-1,$$

with equality if and only if M is Riemannian hyperbolic.

Corollary 2. *Let $M=\Omega/\Gamma$ be compact. Then the following propositions are equivalent:*

1. *M is Riemannian hyperbolic;*
2. *the boundary $\partial\Omega$ is $C^{1+\epsilon}$ for any $0<\epsilon<1$;*
3. *the group Γ is Zariski-dense in $SL(n+1,\mathbb{R})$;*

(4) *the parallel transport along geodesics on* Ω *is an isometry;*
(5) *the Lyapunov exponents of* φ^t *are* -1, 0 *and* 1*;*
(6) φ^t *admits an absolutely continuous measure;*
(7) *the topological entropy satisfies* $h_{top} = n-1$.

5.2. **Noncompact manifolds.** When we talk about global behaviour of the geodesic flow, we actually want to talk about its recurrence properties. The important points are thus those which come back close to their original position infinitely often, in the past and the future. The set of such points is called the nonwandering set of the flow, which is a closed subset of HM.

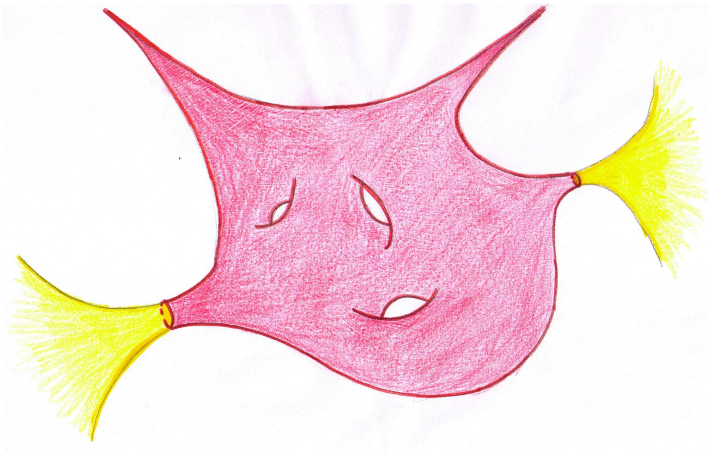


FIGURE 2. The convex core of a geometrically finite surface

For a compact quotient $M = \Omega/\Gamma$, the nonwandering set is the whole of HM; but in general, this is only a small part N of HM, and its projection $\pi(N)$ on M is contained in a certain proper part of the manifold M, called its convex core. Thus, we cannot expect to derive the whole of the geometry from the reccurence properties of the geodesic flow as in theorem 2. Nevertheless, we could expect the same kind of results about the geometry of the convex core.

There are first two different studies to make:

- understand the geometry of noncompact quotients of a given Hilbert geometry (Ω, d_Ω);
- develop general tools to study their geodesic flow, "like the ones we used for compact quotients".

These two studies are to be made in parallel, the first one giving examples for the results of the second one to be applied.

5.2.1. *General dynamical results.* When the manifold is noncompact, the geodesic flow can still admit some *interesting* invariant probability measures, and from a measurable point of view, the space then appears like a *finite* (compact) space.
By an *interesting* measure, we mean a measure which is well adapted to describe the recurrence properties of the flow. For topologically mixing Anosov flows, like the geodesic flow for compact quotients, the Bowen-Margulis measure is *the* measure related to topological entropy: this is the unique measure which achieves the supremum in the variational principle

$$h_{top} = \sup\{h_\mu,\ \mu \text{ invariant probability measure}\}.$$

The measure-theoretical entropy h_μ is still a measure of the complexity of the flow, but from a measurable point a view, in the sense that sizes are considered with respect to the measure μ under consideration, and not with respect to some distance on the space. The variational principle asserts that the dynamical system is always "more chaotic" from a topological point of view and, if there exists a measure of maximal entropy, that is a measure that achieves the supremum, then this measure is in some sense adapted to describe the topological complexity of the flow. It can be a striking fact that such a measure is often not absolutely continuous (with respect to some arbitrary volume form on the manifold), even in the case there exists an invariant volume. For example, for the geodesic flow of a metric of negative curvature on a compact surface, the Bowen-Margulis measure is not absolutely continuous, unless the curvature is constant.

There is a construction of the Bowen-Margulis measure based on Patterson-Sullivan densities, which is a family of measures $(\mu_x)_{x\in\Omega}$ on the boundary $\partial\Omega$. This construction still works when considering noncompact quotients, giving a way of defining such interesting invariant measures for any quotient. However, this process can in some cases yield an infinite measure and also more than one measure, since Patterson-Sullivan densities are in general not unique. Some general dynamical results can be proved when the resulting measure μ_{BM} is a probability measure, in which case it is unique and ergodic.

Theorem 3. *Let $M=\Omega/\Gamma$ be any quotient manifold. If*

(1) *there exists a probability Bowen-Margulis measure μ_{BM} on HM;*
(2) *the geodesic flow has no zero Lyapunov exponent on the nonwandering set;*

then μ_{BM} is the unique measure of maximal entropy and

$$h_{top} = h_{\mu_{BM}} = \delta_\Gamma.$$

To estimate entropies, Ruelle inequality is a first step, and that is the one I used to establish the upper bound in theorem 1. This inequality relates entropy with the sum of positive Lyapunov exponents χ^+, which measures the asymptotic effect of the flow on volumes in the unstable manifolds. The equality in theorem 1 was deduced from the case of equality in Ruelle inequality, which had been clarified by Ledrappier and Young. I extended both results for *Gromov-hyperbolic* Hilbert geometries.

Theorem 4. *Let (Ω, d_Ω) be a* Gromov-hyperbolic *Hilbert geometry and $M=\Omega/\Gamma$ a quotient manifold. For any φ^t-invariant probability measure m, we have*

$$h_m \leqslant \int \chi^+ \, dm,$$

with equality if and only if m has absolutely continuous conditional measures on unstable manifolds.

The assumption of Gromov-hyperbolicity is mainly a technical one to make the proof simpler and there is no doubt it can be removed.

5.2.2. *Geometrically finite surfaces.* No many examples of noncompact manifolds modeled on Hilbert geometries are known yet. I began to work on this question with Ludovic Marquis, and in my thesis, I give the case of surfaces as a first example. Most of the results are inspired by those known for hyperbolic or negatively curved spaces.

In negative curvature, the simplest examples of noncompact manifolds are the so-called geometrically finite manifolds. For Hilbert geometries, such quotients can be defined in the same way, but the more important and difficult part is to understand their properties at infinity. Indeed, in negative curvature, the asymptotic geometry in the cuspidal parts appears to be crucial in various questions. Hilbert geometries are quite rigid compared to negatively-curved spaces and we can expect to control the geometry of the cusps. This is not achieved yet, but we expect the geometry in the cusps to be “asymptotically hyperbolic”: far enough in the cuspidal part, the Hilbert metric would be bi-Lipschitz equality to a hyperbolic metric.

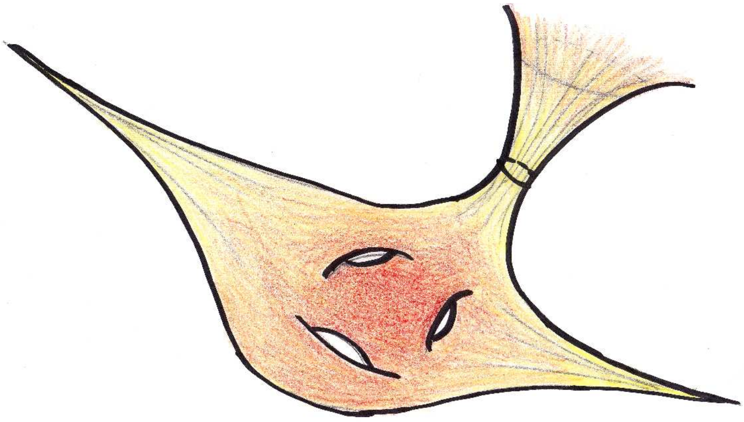


FIGURE 3. A geometrically finite surface

To prove such a result, one has to understand parabolic subgroups of Hilbert geometries, since the fundamental group of a cusp is represented by a parabolic subgroup. This is easy in dimension 2 since isometries of Hilbert geometries are well classified and understood in terms of matrices. So we get the following description of geometrically finite surfaces, that we aim to extend to higher dimensions. Recall that the convex core $C(M)$ of the manifold $M = \Omega/\Gamma$ is the most important part of the manifold for us because it carries the recurrent part of the geodesic flow.

Theorem 5. *Let $M = \Omega/\Gamma$ be a geometrically finite surface. For any $C > 1$, there exists a decomposition of the convex core $C(M)$ into*

$$C(M) = K \bigsqcup \sqcup_{i=1}^{l} C_i,$$

consisting of a compact part K and a finite number of cusps C_i, which carry hyperbolic metrics $\mathtt{h}_i$ and $\mathtt{h}'_i$ such that

- *F, $\mathtt{h}_i$ and $\mathtt{h}'_i$ have the same geodesics on C_i, up to parametrization*
- $\frac{1}{C}\mathtt{h}_i \leqslant \mathtt{h}'_i \leqslant F \leqslant \mathtt{h}_i \leqslant C\mathtt{h}'_i$

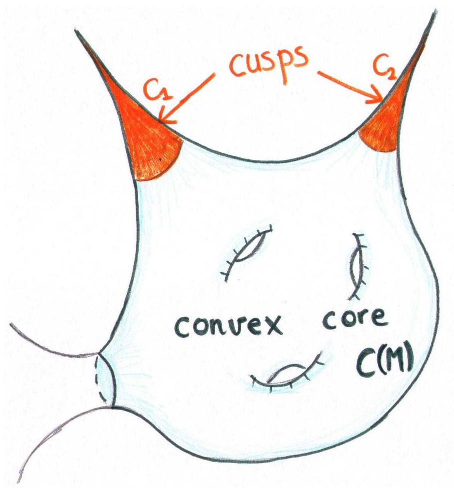


FIGURE 4. Geometry of a geometrically finite surface

Geometrically finite surfaces are then easy to work with: they consist of a compact part, a finite number of cusps, and a finite number of "trumpets" (see figure 5.2.2); and for what concerns the geodesic flow, we are either in a compact part, or in a part where the geometry is entirely controlled, as precisely as we want.

From this, one can derive various results whose proofs are inspired by the Riemannian cases.

Theorem 6. *Let $M = \Omega/\Gamma$ be a geometrically finite surface. Then*

- *the geodesic flow of the Hilbert metric is uniformly hyperbolic on the nonwandering set; in particular, it has no zero Lyapunov exponent.*
- *there exists a finite Bowen-Margulis measure.*

This theorem implies in particular that the general dynamical results are relevant for geometrically finite surfaces. We then deduce an entropy rigidity theorem for finite volume surfaces, which are those geometrically finite surfaces without trumpets, so that the convex core is still the entire manifold.

Theorem 7. *Let $M = \Omega/\Gamma$ be a finite volume surface. Then the topological entropy h_{top} of the geodesic flow on HM satisfies*

$$h_{top} \leqslant 1,$$

with equality if and only if M is Riemannian hyperbolic.

5.3. **Volume entropy.** The volume entropy $h_{vol}(\Omega)$ of a Hilbert geometry (Ω, d_Ω) measures the asymptotic growth rate of volumes of metric balls in (Ω, d_Ω):

$$h_{vol}(\Omega) = \limsup_{R \to +\infty} \frac{1}{R} \log vol(B(o, R)).$$

When Ω admits a compact quotient, an easy adaptation of a theorem of Manning proves that $h_{vol} = h_{top}$. For finite volume manifolds, this equality can fail in negative curvature and this depends heavily on the asymptotic geometry in the cusps. In dimension 2, theorem 6 allows to prove that equality always occurs in Hilbert geometry:

Theorem 8. *Let $M = \Omega/\Gamma$ be a compact manifold or a finite volume surface. Then*

$$h_{top} = h_{vol}(\Omega).$$

The entropy rigidity results then yields the following

Corollary 9. *Let $\Omega \subset \mathbb{RP}^n$ such that*

- *either Ω admits a compact quotient,*
- *or $n = 2$ and Ω admits a finite volume quotient.*

Then $h_{vol}(\Omega) \leqslant n - 1$, with equality if and only if Ω is an ellipsoid.