I have made comments on this text on April, 19, 2013, because some statements or proofs were discovered to be wrong ! I don't know how to fix most of them...

In red, the things which are wrong with comments !
In blue, the things which are partly wrong, that is, not in the stated generality.
In green, some further comments.

À toi, sans qui tout cela ne vaudrait rien.

- *Would you tell me please, which way I ought to go from here ?*
- *That depends a good deal on where you want to get to, said the Cat.*
- *I don't much care where..., said Alice.*
- *Then it doesn't matter which way you go, said the Cat.*
- *...so long as I get somewhere... Alice added as an explanation.*
- *Oh, you're sure to do that, said the Cat, if only you walk long enough.*

Lewis Carroll, Alice's Adventures in Wonderland.

Introduction

As for most mathematical texts, the organization of this thesis does not reflect the fundamentally anarchic process of research. It is written in such a way that one can read it from the beginning to the end, with all the arguments and details coming up at the suitable logical moment for the reader to be convinced. This approach though coherent and rigorous is not always the best way to help the reader in his understanding.

In this introduction, I would like to present the main results of this work as they showed up all along the last three years, with emphasis on motivations and informal logical links. I hope that will provide a good entrance point into the thesis.

Take an abstract smooth compact manifold M, which admits a hyperbolic structure M_0, that is, a metric of constant sectional curvature -1. M_0 can be seen as the quotient $\mathbb{H}/\Gamma_0$ in the Beltrami model of the hyperbolic space: the space $\mathbb{H}$ is the unit ball Ω_0 in $\mathbb{R}^n \subset \mathbb{RP}^n$ with the distance between two distinct points x and y being defined by

$$d(x, y) = \frac{1}{2} \log[a, b, x, y], \tag{1}$$

where a and b are the two intersection points of the line (xy) with the boundary $\partial\Omega_0$ of Ω_0 (see figure 1); the full group of isometries of $\mathbb{H}$ is the group $PO(n, 1)$ and Γ_0 is a discrete subgroup of it. The geodesics on M_0 are just the projections of the lines intersecting Ω_0.

It is sometimes possible to deform continuously and in a non-trivial way the group Γ_0 into discrete groups $\Gamma_t < PGL(n + 1, \mathbb{R})$. In other words the representation Γ_0 of the fundamental group of M in $PO(n, 1)$ is deformed into representations Γ_t in $PGL(n + 1, \mathbb{R})$; continuity is considered with respect to the compact-open topology, and by non-trivial one means that Γ_t is not a subgroup of some conjugate of $PO(n, 1)$. A theorem of Koszul [48] affirms that, at least for small t, there exist corresponding deformations of the ball Ω_0 into bounded convex sets $\Omega_t \subset \mathbb{R}^n$ such that Γ_t still acts on Ω_t; the quotient $M_t = \Omega_t/\Gamma_t$ is a **convex projective** structure on M.

In full generality, a convex projective structure is a pair (Ω, Γ) consisting of a convex proper open subset Ω of $\mathbb{RP}^n$ and a representation of $\pi_1(M)$ as a discrete group $\Gamma < PGL(n + 1, \mathbb{R})$, such that Γ acts on Ω and Ω/Γ is diffeomorphic to M. Two such structures Ω/Γ and Ω'/Γ' are equivalent if the quotients are equivalent as projective manifolds: there is a projective transformation γ such that $\gamma.\Omega = \Omega'$ and $\Gamma' = \gamma\Gamma\gamma^{-1}$.

The deformation that was just considered is thus a deformation of the hyperbolic structure M_0

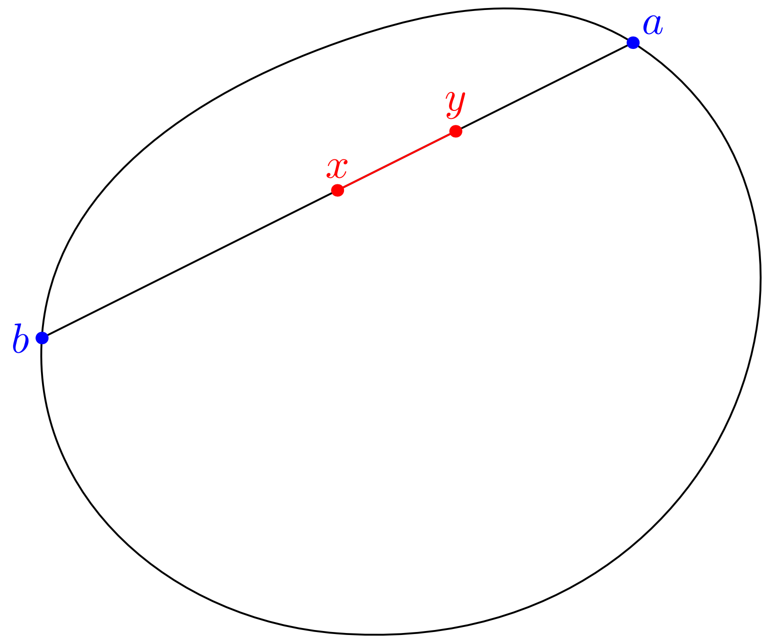

Figure 1: The Beltrami-Hilbert distance

into non-equivalent convex projective ones M_t. Formula (1) defines a metric on each Ω_t, called the Hilbert metric of Ω_t, whose geodesics are still the lines; since the metric is defined by a cross-ratio, it is projectively invariant and thus gives a metric on each M_t. The non-triviality of the deformation implies that the manifold M_t is not isometric to M_0.

The existence of such deformations was a long standing question. The first examples of non-hyperbolic strictly convex projective manifolds were given by Kac and Vinberg in 1967 [43], and explicit deformations of hyperbolic structures were constructed in any dimension in 1984 by Johnson and Millson [42]. A major paper in this story is certainly [35]. Goldman provides there an acute study of convex projective compact surfaces. Among other things, he shows that the set $\mathcal{G}(\Sigma_g)$ of all convex projective structures on the surface Σ_g of genus $g \geqslant 2$, considered up to equivalence, is a smooth manifold diffeomorphic to $\mathbb{R}^{16g-16}$. The space $\mathcal{G}(\Sigma_g)$ contains the Teichmüller space $\mathcal{T}(\Sigma_g)$ of non-equivalent hyperbolic structures as a submanifold of dimension $6g-6$, hence proving that convex projective structures are much more general than hyperbolic ones. In fact, Choi and Goldman [20] went further: they showed that $\mathcal{G}(\Sigma_g)$ was exactly the connected component of $\mathcal{T}(\Sigma_g)$ in the set of faithful and discrete representations of the fundamental group $\pi_1(\Sigma_g)$ in $PGL(3,\mathbb{R})$, up to conjugation. This study was extended by the same authors to 2-orbifolds [21].

The general question about these convex projective deformations of hyperbolic structures is: which properties of hyperbolic manifolds stay true after deformation, which ones are lost ? In particular, do some of them characterize hyperbolic structures among convex projective ones ? These could be metric properties, geometric properties, group properties... For example, after deformation, the Hilbert metric is not a Riemannian metric anymore, it is only Finslerian: instead of having a scalar product on each tangent space $T_x\Omega$, we have a norm $F(x,.)$. On the other side, the following fundamental result implies in particular that some amount of hyperbolicity remains after deformation of a hyperbolic manifold.

Theorem (Benoist, [7])**.** *Let $M = \Omega/\Gamma$ be a convex projective compact manifold. The following propositions are equivalent:*

- *Ω is strictly convex;*
- *the boundary $\partial\Omega$ of Ω is C^1;*
- *the space (Ω, d_Ω) is Gromov-hyperbolic;*
- *Γ is Gromov-hyperbolic.*

In this thesis, I am interested in dynamical properties of the geodesic flow of the Hilbert metric, whose study was initiated by Yves Benoist in [7]. The geodesic flow φ^t of the Hilbert metric on a convex projective manifold M is defined on the homogeneous tangent bundle $HM = TM \smallsetminus \{0\}/\mathbb{R}_+$: given a pair $w = (x, [\xi])$ consisting of a point $x \in M$ and a direction $[\xi] \in H_xM$, follow the geodesic leaving x in the direction $[\xi]$ during the time t. On $H\Omega$, the picture is easy to see: one follows the lines at unit speed...

Then how does the dynamics of the geodesic flow of the metric change when the structure M_0 is deformed into M_t ? Yves Benoist proved in [7] that it is still an Anosov flow, and the question I was first asked to answer was: does its topological entropy change ?
Topological entropy is a major invariant in the theory of dynamical systems which roughly speaking measures how the system separates the points, how much it is chaotic. (See section 1.6 for the formal definition.) An answer is provided by the following

Theorem 1. *Let $M = \Omega/\Gamma$ be a strictly convex projective compact manifold of dimension n. Its topological entropy h_{top} satisfies the inequality*

$$h_{top} \leqslant (n-1),$$

with equality if and only if M is Riemannian hyperbolic.

$n-1$ is the topological entropy of the hyperbolic geodesic flow, so this theorem asserts in particular that a non-trivial deformation of a hyperbolic structure makes the topological entropy decrease. This is a surprising fact when one thinks of the famous result of Besson, Courtois and Gallot ([12, 13]) which says that, if one makes vary the curvature of M_0 without changing the volume, the topological entropy has to increase. I did not find any satisfying explanation for this phenomenon : is there some volume involved that would increase during the deformation ? is there a renormalization of the geometries that would make the entropy stay constant, or increase ?
I then turned to look at how the entropy could vary: given the hyperbolic structure M_0, can we make the topological entropy decrease as much as we want by deforming M_0 into the convex projective world ? For instance, consider the space $\mathcal{G}(\Sigma_g)$ defined above, of all convex projective structures on the surface Σ_g, up to equivalence. It is not difficult to see that the entropy function $h_{top} : \mathcal{G}(\Sigma_g) \longrightarrow (0,1]$ is a continuous map (section 5.4); its image is then a sub-interval of $(0,1]$,

and the question is: is it surjective ? I first hoped to understand compactifications of $\mathcal{G}(\Sigma_g)$ and to interpretate boundary points, with the assumption that the infimum should be attained on the boundary of $\mathcal{G}(\Sigma_g)$. I did not dig deep enough to know if it was a good intuition. Very recently, Xin Nie [59] showed how to make the entropy decrease to 0 in the Kac-Vinberg examples, in dimension 2, 3 and 4.

At the very moment I was wandering within these considerations, Ludovic Marquis was beginning the study of convex projective manifolds of finite volume ([57, 55]). I thus thought about extending theorem 1 to finite volume manifolds.
I then had to look back at the proof of theorem 1. The fundamental tools I used for the inequality can be summarized by the formula

$$h_{top} = h_{\mu_{BM}} \leqslant \int_{HM} \chi^+ \; d\mu_{BM}. \tag{2}$$

Explaining this formula will shed some light on the problems I had to face with.
Given an invariant probability measure μ of a dynamical system, one can define the entropy h_μ of this measure. As topological entropy, this is an indicator of the complexity of the system, but from a measure point of view: "sizes" are considered with respect to μ and not with respect to a certain distance d. (See section 4.1.2 for formal definitions.)
The variational principle makes a link between measure-theoretic and topological entropies: it asserts that topological entropy is the supremum of the entropies of all invariant probability measures of the system: $h_{top} = \sup_\mu h_\mu$. A natural question is to know if there exists some measure that achieves this maximum.
The measure μ_{BM} appearing in equation (2) is the unique measure of maximal entropy of the geodesic flow on HM. BM stands for Bowen and Margulis who gave two independent constructions of it ([15, 16], [52, 53]), which is now known as the Bowen-Margulis measure. It is defined for geodesic flows of compact Riemannian manifolds of negative curvature, or more generally for topologically mixing Anosov flows [47], and is in any case the unique measure of maximal entropy.

The inequality

$$h_\mu \leqslant \int_W \chi^+ \; d\mu$$

is the general Ruelle inequality [70], which is valid for any invariant probability measure μ of a C^1 flow on a compact manifold W. In this formula, χ^+ is the sum of positive Lyapunov exponents, which is equal μ-almost everywhere to the asymptotic expansion by φ^t of volumes in unstable manifolds:

$$\chi^+ = \lim_{t\to+\infty} \frac{1}{t} \log|\det d\varphi^t|.$$

Pesin [65] proved that equality occurs if μ is absolutely continuous, and Ledrappier and Young [49] proved that equality occurs if and only if μ has absolutely continuous conditional measures on unstable manifolds. This last statement is used to prove the equality case in theorem 1: indeed, Benoist had already observed in [7] that there could not be an absolutely continuous invariant measure unless the structure was hyperbolic.

The main task was then to write down such an equation for some noncompact convex projective manifolds.

Topological entropy has a natural generalization to dynamical systems in noncompact spaces, proposed by Bowen [17], and for which Handel and Kitchens [39] proved a variational principle under very general assumptions.
The Bowen-Margulis measure has also a generalization for noncompact negatively curved Riemannian manifolds, which is based on Sullivan's construction [72] for hyperbolic spaces. It makes use of Patterson-Sullivan measures, which are measures defined geometrically on the boundary at infinity of the universal cover. A lot of attention has been paid to these measures, that provide bridges between geometry and dynamics. Roblin's version of Hopf-Tsuji-Sullivan theorem (theorem 1.7 in [67]) is the most achieved version of what is known about them (see theorem 4.2.4).
All of this makes sense in the context of Hilbert geometries, at least when the geometry exhibits some hyperbolic behaviour. In this thesis, this means the Hilbert geometry is defined by a strictly convex set with C^1 boundary; for example, it includes all the Hilbert geometries which are Gromov-hyperbolic (see sections 1.1.3 and 1.1.4).

If the Bowen-Margulis measure can always be defined on HM, its behaviour and properties are not always easy to determine. In [67], Roblin showed that lots of dynamical results could be derived from the only fact that the Bowen-Margulis measure is finite. Obviously, equation (2) could not make sense in the case μ_{BM} is not finite. In the context of pinched negatively curved manifolds, Otal and Peigné [61] proved that, under this finiteness hypothesis, μ_{BM} was indeed the only measure of maximal entropy, hence generalizing what was known for compact quotients. In fact, they proved an even stronger result:

Theorem (Otal-Peigné [61])**.** *Let X be a simply connected Riemannian manifold of pinched negative curvature, and $M = X/\Gamma$ any quotient manifold, where Γ is a discrete subgroup of isometries of X. Then*

- *the topological entropy h_{top} of the geodesic flow on HM satisfies $h_{top} = \delta_\Gamma$;*
- *if there is some probability Bowen-Margulis measure μ_{BM}, then it is the unique measure of maximal entropy; otherwise, there is no measure of maximal entropy.*

Here δ_Γ denotes the critical exponent of the group Γ acting on X, which is closely related to Patterson-Sullivan measures :

$$\delta_\Gamma = \limsup_{R\to+\infty} \frac{1}{R} \log N_\Gamma(o, R),$$

where $N_\Gamma(o, R)$ is the number of points of the orbit $\Gamma.o$ of a point o in X under Γ in the metric ball of radius R in X. The equality $h_{top} = \delta_\Gamma$ was already known by Manning [51] for compact quotients.

In chapter 5, I prove the following version of this theorem for quotients of Hilbert geometries :

Theorem 2 (Section 5.1)**.** *Let $M = \Omega/\Gamma$ be the quotient manifold of a strictly convex set Ω with C^1 boundary. Assume there exists a probability Bowen-Margulis measure μ_{BM} on HM. If the geodesic flow has no zero Lyapunov exponent on the nonwandering set, then μ_{BM} is the unique measure of maximal entropy and*

$$h_{top} = h_{\mu_{BM}} = \delta_\Gamma.$$

The proof of this result is inspired though simplified from the one of [61], which is itself based on technics developed around 1980 in the study of non-uniformly hyperbolic systems; the already mentioned paper [49] of Ledrappier and Young is one of the most famous illustrations of these technics. By adapting them to Hilbert geometries, Pesin-Ruelle inequality and its case of equality appeared then as (almost) direct consequences in the case of Gromov-hyperbolic Hilbert geometries :

Theorem 3 (Section 5.2)**.** *Let (Ω, d_Ω) be a Gromov-hyperbolic Hilbert geometry and $M = \Omega/\Gamma$ a quotient manifold. For any φ^t-invariant probability measure m, we have*

$$h_\mu \leqslant \int \chi^+ \, dm,$$

with equality if and only if m has absolutely continuous conditional measures on unstable manifolds.

(See the text for an explanation of the mistake.)

Here is time to make a break to reveal the point of view, kept hidden until now, that allowed me to prove theorem 1 and to extend the above mentioned technics. This point of view is the one Patrick Foulon developed in [33] to study second-order differential equations. Geodesic flows of usual regular Finsler metrics are special cases where Foulon's dynamical formalism can be applied. In section 2.1, I extend this formalism in the context of Hilbert geometries defined by (strictly) convex sets with C^1 boundary; the flatness of the geometries is crucial here to deal with less regular metrics. In particular, it allows me to define a parallel transport along geodesics that indeed contains all the informations about the asymptotic dynamics along this geodesic. For example, the Anosov property for the geodesic flow on compact quotients, proved by Benoist, can be seen as a direct consequence of this observation.

A striking and crucial fact is that this parallel transport is in general not an isometry, and that is what makes the geodesic flow have a different behaviour than in Riemannian spaces. In particular, the sum χ^+ of positive Lyapunov exponents can be expressed (along a regular orbit) as

$$\chi^+ = (n-1) + \eta,$$

where

$$\eta = \lim_{t \to +\infty} \frac{1}{t} \log |\det T^t|$$

represents the effect of the parallel transport T^t on volumes. Theorem 1 now becomes an easy corollary of this and equation (2): we get

$$h_{top} \leqslant (n-1) + \int_{HM} \eta \, d\mu_{BM},$$

and $\displaystyle\int_{HM} \eta \, d\mu_{BM} = 0$ for simple reasons of symmetry (see the proof of proposition 5.3.1).

While working on theorem 1, I had noticed that one could read the Lyapunov exponents of a given geodesic on the shape of the boundary $\partial\Omega$ of Ω at the endpoint of the geodesic (see proposition 5.4 in [25]). Chapter 3 is dedicated to generalize this remark to any Hilbert geometry defined by

a strictly convex set with C^1 boundary. It relates Lyapunov exponents, parallel transport and the shape of the boundary $\partial\Omega$.
As a consequence of that, I show in section 3.5 how Lyapunov manifolds tangent to the various subspaces in Lyapunov-Osedelets decomposition can be easily defined. The flatness of the geometry appears to be essential in this construction, so I do not know if a similar thing could be expected in the case of Riemannian manifolds of negative curvature, or for general Anosov flows.

At the same time I was considering these general questions, I was also looking for some specific quotients theorem 2 could be applied to.
The only examples that were available then were the finite volume surfaces studied by Ludovic Marquis in [57]. For what I was concerned with, the important fact was that such a surface could be decomposed into a compact part and a finite number of cusps, whose geometry was well understood. In fact, one can easily see from [57] that the Hilbert metric in a cusp is bi-Lipschitz equivalent to a Riemannian hyperbolic metric. This simple observation suffices to prove that the geodesic flow is uniformly hyperbolic, hence has no zero Lyapunov exponent, and to adapt proofs used in hyperbolic geometry to get the finiteness of the Bowen-Margulis measure. The proof of theorem 1 then readily applies to this situation: $\displaystyle\int \eta \ d_{\mu_{BM}} = 0$ just comes from the symmetry of the Bowen-Margulis measure, which is a very general fact; as for the equality case, Benoist's argument in [7] still gives that there is no invariant absolutely continuous measure, unless the structure is hyperbolic. Then we get

Theorem 4 (Theorem 5.3.6)**.** *Let $M = \Omega/\Gamma$ be a surface of finite volume. Then*

$$h_{top} \leqslant 1,$$

with equality if and only if M is Riemannian hyperbolic.

(I can't get this result since the proof of the previous theorem does not work.)

The last arguments convinced me that the crucial property was the decomposition of the manifold into a compact part and a controllable part, which was enough to extend the methods used in hyperbolic geometry. Since essentially nothing more than Marquis' results was known yet about the geometry of noncompact quotients, I turned my mind to hyperbolic geometry, looking for possible extensions to higher dimensions and more general quotients.

In hyperbolic geometry, there is a natural generalization of finite volume manifolds, which are geometrically finite manifolds. In those manifolds, the convex core, which is known to carry the essential part of the dynamics, has finite volume. Then, together with Ludovic Marquis [26], we began to investigate the notion of geometrically finite quotients of Hilbert geometries.
Let us remark that, if this notion of geometrical finiteness has become classical now, it was not the case until Bowditch [14] clearly stated several equivalent definitions of it. In the context of a strictly convex set with C^1 boundary, the characterization by the limit set seemed to be a good point of departure, and we adopted it; see definition 1.4.3. The study of such quotients is still on progress. The only general result we were able to prove at the moment is the following

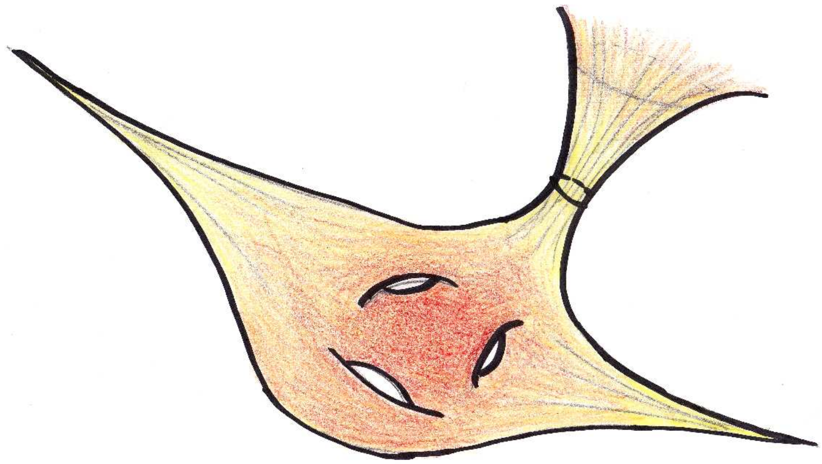

Figure 2: A geometrically finite surface

Theorem 5 ([26] and theorem 1.4.8)**.** *Let $M = \Omega/\Gamma$ be a geometrically finite manifold. Then the convex core of M can be decomposed as a compact part and a finite number of cusps.*

(This result is not true in this generality, as we discovered later; see our paper [26]; however, it is true in dimension 2.)

But this is not enough to make all the things work, especially about the dynamics, because some of the technics failed without any geometric control in the cuspidal parts of the manifolds. We thought at some moment to have proved that cusps had essentially the same geometry as in hyperbolic manifolds, but there was an important mistake in our approach. In this thesis, I provide a description of what occurs in dimension 2, which is based on Marquis' work [57]. The main results about dynamics on geometrically finite surfaces are summarized in the following

Theorem 6. *Let $M = \Omega/\Gamma$ be a geometrically finite surface. Then*

- *the geodesic flow of the Hilbert metric is uniformly hyperbolic on the nonwandering set (theorem 2.5.2); in particular, it has no zero Lyapunov exponent;*
- *there exists a finite Bowen-Margulis measure (Section 4.3).*

This shows that geometrically finite surfaces satisfy the hypotheses of theorem 2. The technics I use to study these noncompact surfaces are classical and only depend on the understanding of the asymptotic geometry of the cusps. For example, these technics work automatically for the only available examples of finite volume manifolds of higher dimensions that were constructed by Marquis in [56].

In fact, this control of the geometry in the cusps was already shown to be important in the context of negatively-curved Riemannian manifolds. [28] is a good example of what can happen: in this article, Dal'bo, Otal and Peigné are able, among other things, to construct geometrically finite manifolds of pinched negative curvature whose Bowen-Margulis measure is infinite, and even not ergodic. In [29], Dal'bo, Peigné, Picaud and Sambusetti show that this asymptotic geometry also has a significant effect on volume entropy. The volume entropy h_{vol} of a Riemannian manifold (M, g) measures the asymptotic exponential growth of volume of metric balls in the universal cover $\widetilde{M}$:

$$h_{vol} = \limsup_{R \to +\infty} \log vol_g B(o, R),$$

where $B(o, R)$ is the metric ball of radius R in $\tilde{M}$ about an arbitrary point o.
If M is compact and negatively-curved, Manning [51] proved volume and topological entropies coincide; its proof extends without difficulty to Hilbert geometry (proposition 1.6.2). But this becomes false for finite volume manifolds, and depends heavily on the geometry of the cusps:

Theorem (Dal'bo, Peigné, Picaud, Sambusetti [29])**.**

- *Let M be a negatively-curved Riemannian manifold of finite volume. If M is asymptotically $1/4$-pinched, then $h_{vol} = h_{top}$.*
- *For any $\epsilon > 0$, there exists a finite volume $(1/4 + \epsilon)$-pinched manifold such that $h_{top} < h_{vol}$.*

In Hilbert geometry, I guess we cannot build such counter-examples. Once again, this depends on our understanding of the cusps. As it could be expected, nothing like this can happen for surfaces:

Theorem 7 (Section 4.4)**.** *Let $M = \Omega/\Gamma$ be a surface of finite volume. Then*

$$h_{vol} = h_{top} = \delta_\Gamma.$$

All of this admits the following corollaries about volume entropy of some Hilbert geometries.

Corollary 8 (Corollary 5.3.5)**.** *Let $\Omega \subset \mathbb{RP}^n$ be a strictly convex proper open set which admits a compact quotient. Then its volume entropy h_{vol} satisfies*

$$h_{vol} \leqslant n - 1,$$

with equality if and only if Ω is an ellipsoid.

Corollary 9 (Corollary 5.3.7)**.** *Consider the Hilbert geometry defined by a strictly convex proper open subset Ω of $\mathbb{RP}^2$ with C^1 boundary which admits a quotient of finite volume. Then its volume entropy h_{vol} satisfies*

$$h_{vol} \leqslant 1,$$

with equality if and only if Ω is an ellipse.

It is conjectured that the volume entropy of an arbitrary Hilbert geometry is always smaller than $n - 1$. This conjecture was shown to be true in dimension 2 by Berck, Bernig and Vernicos [10], who also proved that $h_{vol} = n - 1$ if the convex set had $C^{1,1}$ boundary. The last two corollaries confirm this conjecture for some specific classes of Hilbert geometries, providing also an infinite class of examples whose volume entropy is strictly between 0 and $n - 1$.

Let me end this introduction by describing the contents of each chapter.

Chapter 1 first provides a short introduction to Hilbert geometries and recalls some already known notions and results. Quotients of Hilbert geometries are studied in sections 1.3 and 1.4. Some of the new geometrical results inspired from [26] are given here: section 1.3 describes the parabolic subgroups and the geometry of cusps; section 1.4.3 defines geometrically finite manifolds and their convex core is decomposed in theorem 1.4.8; we focus on surfaces in section 1.4.4.

Chapter 2 begins the study of the geodesic flow of Hilbert metrics. The first thing is to extend Foulon's dynamical formalism. We then show that it provides a good tool in Hilbert geometries; the fundamental results are propositions 2.4.1 and 2.4.5. Section 2.5 ends this chapter by proving the uniform hyperbolicity of the geodesic flow on compact quotients and on geometrically finite surfaces.

In chapter 3, we get interested in Lyapunov exponents of the geodesic flow. We show in particular that Oseledets' theorem can be applied to any quotient manifold. In section 3.4, we explain the links between parallel transport, Lyapunov exponents and the shape of the boundary at infinity. For this, we need to introduce a new regularity property of convex functions. Some time is spent on this property, that we especially show to be projectively invariant and thus adapted to our setting. As a consequence, we can easily define in section 3.5 Lyapunov manifolds tangent to the Lyapunov-Oseledets filtrations.

Chapter 4 studies the properties of Patterson-Sullivan and Bowen-Margulis measures. We first explain why some general theorems known for Riemannian manifolds of negative curvature, especially theorem 4.2.4 remain true in our context. Section 4.3 proves that any Bowen-Margulis measure of a geometrically finite surface is finite. In the last section, we show that critical exponent and volume entropy coincide on a surface of finite volume.

In the last chapter, we first recall how to construct measurable partitions which allow to effectively compute entropies and one applies it to get theorem 2. Ruelle inequality and its case of equality are then extended to some noncompact quotients. As a consequence, one gets theorems 1 and 4 and their corollaries about volume entropy.

Présentation

Comme c'est le cas pour la plupart des textes mathématiques, l'organisation de cette thèse ne reflète pas le processus fondamentalement anarchique de la recherche. Elle est pensée de telle façon qu'on puisse la lire linéairement d'un bout à l'autre, les divers arguments étant donnés aux moments les plus "logiques". Cette approche, bien que cohérente et rigoureuse, n'est cependant pas toujours la plus adaptée à la compréhension du lecteur.
Dans cette introduction, j'aimerais présenter les résultats de mon travail tels qu'ils sont apparus au cours de ces trois ans, en insistant sur les motivations et les liens informels qui les unissent. J'espère que cela permettra d'entrer plus facilement dans la thèse.

Soit M une variété lisse abstraite, supposée compacte, qui admet une structure hyperbolique M_0, c'est-à-dire une métrique à courbure négative constante égale à -1. M_0 peut être vue comme le quotient $\mathbb{H}/\Gamma_0$ dans le modèle de Beltrami de l'espace hyperbolique: l'espace $\mathbb{H}$ est la boule unité Ω_0 de $\mathbb{R}^n \subset \mathbb{RP}^n$ et la distance entre deux points x et y de Ω_0 est définie par

$$d(x,y) = \frac{1}{2}\log[a,b,x,y], \tag{3}$$

où les points a et b sont les points d'intersection de la droite (xy) avec le bord $\partial\Omega_0$ de Ω_0 (c.f. figure 3); le groupe d'isométries de $\mathbb{H}$ est le groupe $PO(n,1)$ et Γ_0 en est un sous-groupe discret, isomorphe au groupe fondamental de M. Les géodésiques de M_0 sont exactement les projections sur M_0 des droites qui intersectent Ω_0.

Il est parfois possible de déformer de façon continue et non triviale le groupe Γ_0 en des groupes discrets $\Gamma_t < PGL(n+1,\mathbb{R})$. Autrement dit, la représentation Γ_0 du groupe fondamental de M dans $PO(n,1)$ est deformée en représentations Γ_t dans $PGL(n+1,\mathbb{R})$; la continuité est entendue au sens de la topologie compacte-ouverte, et non triviale signifie que Γ_t n'est pas conjugué à un sous-groupe de $PO(n,1)$. Un théorème de Koszul [48] affirme, au moins pour t petit, qu'il existe des déformations correspondantes de la boule Ω_0 en convexe borné $\Omega_t \subset \mathbb{R}^n$ sur lequel Γ_t agit; le quotient $M_t = \Omega_t/\Gamma_t$ est une structure **projective convexe** sur M.
En toute généralité, une structure projective convexe est une paire (Ω,Γ) constituée d'un ouvert convexe propre Ω de $\mathbb{RP}^n$ et d'une représentation du groupe fondamental $\pi_1(M)$ en un groupe discret $\Gamma < PGL(n+1,\mathbb{R})$ agissant sur Ω avec quotient Ω/Γ difféomorphe à M. Deux telles structures Ω/Γ et Ω'/Γ' sont dites équivalentes si les quotients sont équivalents en tant que variétés projectives: il existe une transformation projective γ telle que $\gamma.\Omega = \Omega'$ et $\Gamma' = \gamma\Gamma\gamma^{-1}$.

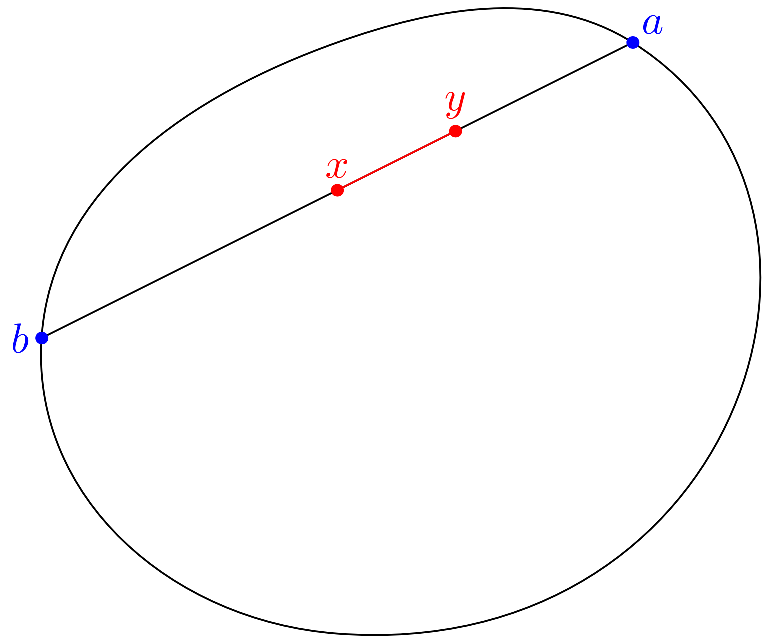

Figure 3: La distance de Beltrami-Hilbert

La déformation considérée ci-dessus apparaît ainsi comme la déformation d'une structure hyperbolique M_0 en structures projective convexes M_t non équivalentes. La formule (3) définit une métrique sur chaque Ω_t, appelée métrique de Hilbert de Ω_t, dont les géodésiques sont encore les droites; comme cette métrique est définie par un birapport, elle est projectivement invariante et donne donc une métrique sur chaque variété quotient M_t. La déformation étant non triviale, M_t n'est pas isométrique à M_0.

L'existence de telles déformations est longtemps restée une question ouverte. Les premiers exemples de variétés projectives strictement convexes furent ceux de Kac et Vinberg en 1967 [43], et un procédé explicite de déformation de structures hyperboliques a été proposé en toute dimension par Johnson et Millson en 1984 [42]. L'article [35] de Goldman constitue une étape fondamentale dans cette histoire. On y trouve une étude approfondie des structures projectives convexes sur les surfaces: entre autres choses, Goldman prouve que l'ensemble $\mathcal{G}(\Sigma_g)$ de toutes les structures projectives convexes sur la surface Σ_g de genre $g \geqslant 2$, à équivalence près, forme une variété lisse difféomorphe à $\mathbb{R}^{16g-16}$. L'espace de Teichmüller $\mathcal{T}(\Sigma_g)$ des structures hyperboliques sur Σ_g à équivalence près, apparaît comme une sous-variété de dimension $6g-6$ de l'espace $\mathcal{G}(\Sigma_g)$, prouvant que les structures projectives convexes sont bien plus souples que les structures hyperboliques. En fait, Choi et Goldman [20] sont allés plus loin en prouvant que $\mathcal{G}(\Sigma_g)$ était exactement la composante connexe de $\mathcal{T}(\Sigma_g)$ dans l'espace des représentations fidèles et discrètes du groupe fondamental $\pi_1(\Sigma_g)$ dans $PGL(3,\mathbb{R})$ à conjugaison près. Les mêmes auteurs ont étendu cette étude au cas des orbifolds de dimension 2 dans [21].

La question générale concernant ces déformations de structures hyperboliques en structures projectives convexes est la suivante: quelles propriétés des variétés hyperboliques sont conservées après déformations, lesquelles sont perdues ? En particulier, certaines d'entre elles permettent-elles de caractériser les structures hyperboliques parmi les structures projectives convexes ? Il peut s'agir, selon les intérêts, de propriétés métriques ou géométriques, de propriétés des groupes en jeu... Par exemple, après déformation, la métrique de Hilbert n'est plus une métrique de Riemann mais seulement une métrique de Finsler: au lieu d'avoir un produit scalaire sur chaque espace tangent $T_x\Omega$, on a une norme $F(x,.)$. D'un autre côté, le résultat fondamental ci-après entraîne en particulier que certaines propriétés de type hyperbolique sont préservées lorsqu'on déforme une variété hyperbolique.

Théorème (Benoist, [7]). *Soit $M = \Omega/\Gamma$ une variété projective convexe compacte. Les propositions suivantes sont équivalentes:*

- Ω *est strictement convexe;*
- *le bord $\partial\Omega$ de Ω est C^1;*
- *l'espace (Ω, d_Ω) est Gromov-hyperbolique;*
- Γ *est Gromov-hyperbolique.*

Dans cette thèse, je me suis intéressé aux propriétés dynamiques du flot géodésique de la métrique de Hilbert, dont l'étude a débuté avec les travaux d'Yves Benoist [7]. Le flot géodésique φ^t de la métrique de Hilbert d'une variété projective convexe M est défini sur le fibré tangent homogène $HM = TM \smallsetminus \{0\}/\mathbb{R}_+$: étant donné un couple $w = (x, [\xi])$ formé d'un point $x \in M$ et d'une direction $[\xi] \in H_xM$, il s'agit de suivre la géodésique partant de x dans le direction $[\xi]$ pendant le temps t. Sur $H\Omega$, ceci est très facile à voir: il s'agit de suivre les droites à vitesse 1...

Comment la dynamique du flot géodésique change t-elle lorsque la variété hyperbolique M_0 est déformée en M_t ? Yves Benoist a montré dans [7] que le flot reste un flot d'Anosov, et la première question à laquelle j'ai cherché à répondre était la suivante: l'entropie topologique varie t-elle ?
L'entropie topologique est un invariant essentiel dans la théorie des systèmes dynamiques qui mesure comment le système "sépare les points", à quel point il est chaotique. (Voir section 1.6 pour une définition formelle.) Le théorème suivant répond à la question:

Théorème 1. *Soit $M = \Omega/\Gamma$ une variété projective strictement convexe, compacte, de dimension n. L'entropie topologique h_{top} du flot géodésique de la métrique de Hilbert de M satisfait à l'inégalité*

$$h_{top} \leqslant (n-1),$$

avec égalité si et seulement si M est riemannienne hyperbolique.

$n-1$ est l'entropie topologique du flot géodésique hyperbolique. Ce théorème montre donc en particulier qu'une déformation non triviale d'une structure hyperbolique fait diminuer l'entropie. C'est un fait assez surprenant lorsqu'on pense au résultat obtenu par Besson, Courtois et Gallot ([12, 13]) qui affirme que, si l'on fait varier la courbure de M_0 sans changer le volume, l'entropie topologique doit augmenter. Je n'ai pas trouvé d'explication raisonnable à cette apparente contradiction: y a t-il un certain volume en jeu qui augmenterait lors de la déformation ? dans ce cas, quel est-il ?

existe t-il une "renormalisation" naturelle qui ferait que l'entropie augmente, ou reste constante ?
J'ai aussi essayé de comprendre les variations de l'entropie: étant donné une structure hyperbolique M_0, peut-on faire tendre l'entropie vers 0 en déformant M_0 dans le monde convexe projectif ? Par exemple, considérons l'espace $\mathcal{G}(\Sigma_g)$, défini ci-dessus, de toutes les structures projectives convexes sur la surface Σ_g, à équivalence près. Il n'est pas difficile de voir que l'entropie $h_{top} : \mathcal{G}(\Sigma_g) \longrightarrow (0,1]$ est une fonction continue (voir section 5.4); son image est donc un sous-intervalle de $(0,1]$, et on peut donc se demander si elle est surjective, ou si elle est propre. J'ai d'abord espéré comprendre les différentes compactifications de $\mathcal{G}(\Sigma_g)$ dans l'idée d'interpréter les points du bord en termes de dynamique, en supposant que l'infimum serait atteint sur le bord de $\mathcal{G}(\Sigma_g)$. Je n'ai pas cherché assez loin pour savoir si cette intuition était bonne. Très récemment, Xin Nie [59] a montré qu'on pouvait faire diminuer l'entropie jusqu'à 0 dans certains exemples de Kac-Vinberg, en dimensions 2, 3 et 4.

Au moment même où j'étais plongé dans ces considérations, Ludovic Marquis commençait à travailler sur les variétés projectives convexes de volume fini ([57, 55]). Je pensais alors étendre le théorème 1 au contexte des variétés de volume fini.
Il fallait regarder de plus près la preuve du théorème 1. Les outils fondamentaux que j'avais utilisés pour prouver l'inégalité se résument essentiellement à la formule:

$$h_{top} = h_{\mu_{BM}} \leqslant \int_{HM} \chi^+ \, d\mu_{BM}. \tag{4}$$

Expliquer cette formule va nous aider à comprendre les problèmes auxquels j'étais alors confronté.
Étant donné une probabilité invariante μ d'un système dynamique, on peut définir son entropie de Kolmogorov h_μ. Tout comme l'entropie topologique, c'est un indicateur de la complexité du système, observé cette fois avec un point de vue mesurable: les "volumes" sont mesurés par la mesure μ et n'ont pas de rapport avec une quelconque distance d. (Voir section 4.1.2 pour des définitions formelles.)
Le principe variationnel fait le lien entre l'entropie topologique et l'entropie de Kolmogorov: ce principe affirme que l'entropie topologique est le supremum des entropies de toutes les probabilités invariantes du système: $h_{top} = \sup_\mu h_\mu$. Un problème naturel est alors de chercher une mesure qui réalise ce supremum.
La mesure μ_{BM} qui apparaît dans l'équation (4) est l'unique mesure d'entropie maximale du flot géodésique sur HM. Les lettres BM font référence à Bowen et Margulis qui ont donné deux constructions indépendantes de cette mesure ([15, 16], [52, 53]), que l'on connaît maintenant sous le nom de mesure de Bowen-Margulis. Elle est définie pour les flots géodésiques de variétés riemanniennes à courbure négative, ou de façon plus générale, pour les flots d'Anosov topologiquement mélangeants; c'est, dans tous les cas, *l'unique* mesure d'entropie maximale.

L'inégalité

$$h_\mu \leqslant \int_W \chi^+ \, d\mu$$

est l'inégalité de Ruelle [70], qui est vérifiée pour toute probabilité invariante μ d'un flot de classe C^1 sur une variété compacte W. Dans cette formule, χ^+ est la somme des exposants de Lyapunov positifs, qui mesure, μ-presque partout, l'effet de φ^t sur les volumes des variétés instables:

$$\chi^+ = \lim_{t\to+\infty} \frac{1}{t} \log|\det d\varphi^t|.$$

Pesin [65] a montré que l'égalité a lieu lorsque μ est absolument continue, et Ledrappier et Young [49] ont montré qu'il y avait égalité si et seulement si la mesure μ avait ses mesures conditionnelles instables absolument continues. Ce dernier résultat est utilisé pour étudier le cas d'égalité dans le théorème 1: en fait, Benoist avait déjà remarqué dans [7] qu'il ne pouvait y avoir de mesure invariante absolument continue, sauf dans le cas d'une structure hyperbolique.

La tâche principale consistait donc à obtenir une telle (in)équation pour des variétés projectives convexes *non compactes*.

L'entropie topologique a une généralisation naturelle aux systèmes dynamiques définis sur des espaces non compacts, proposée par Bowen [17], et pour laquelle Handel et Kitchens [39] ont prouvé un principe variationnel sous des hypothèses très souples.
La mesure de Bowen-Margulis peut aussi être définie pour les variétés non compactes de courbure négative, à partir de la construction de Sullivan [72], à l'origine dans l'espace hyperbolique. Cette construction est basée sur les mesures de Patterson-Sullivan, qui sont définies de façon géométrique sur le bord à l'infini du revêtement universel. Ces mesures ont fait l'objet de beaucoup d'attention et ont permis de faire de nombreux liens entre géométrie et dynamique. La version de Roblin du théorème de Hopf-Tsuji-Sullivan (théorème 1.7 de [67]) est certainement la version la plus aboutie de ce que l'on peut dire en général à leur propos (voir théorème 4.2.4).
Tout cela a aussi un sens dans le contexte des géométries de Hilbert, au moins pour celles qui présente un certain comportement hyperbolique. Dans cette thèse, on entend par là une géométrie de Hilbert définie par un ouvert strictement convexe à bord de classe C^1; par exemple, cela inclut toutes les géométries de Hilbert qui sont hyperboliques au sens de Gromov (voir sections 1.1.3 et 1.1.4).

Si la mesure de Bowen-Margulis peut toujours être définie sur HM, son comportement et ses propriétés ne sont pas toujours faciles à déterminer. Dans [67], Roblin a montré que de nombreux résultats dynamiques pouvaient être déduits du seul fait que la mesure de Bowen-Margulis était de masse totale finie. Bien sûr, le formule (4) n'aurait pas de sens dans le cas où μ_{BM} n'était pas finie. Dans le contexte des variétés à courbure strictement négative pincée, Otal et Peigné [61] ont montré que, sous cette hypothèse de finitude, μ_{BM} était en fait l'unique mesure d'entropie maximale, généralisant ainsi ce qui était connu pour les quotients compacts. En fait, leur résultat est plus fort que cela puisqu'il clarifie aussi le cas où la mesure est infinie:

Théorème (Otal-Peigné [61]). *Soient X une variété riemannienne simplement connexe, de courbure strictement négative pincée et $M = X/\Gamma$ une variété quotient, où Γ est un sous-groupe discret d'isométries de X, sans torsion. Alors*

- *l'entropie topologique h_{top} du flot géodésique sur HM satisfait $h_{top} = \delta_\Gamma$;*
- *s'il existe une mesure de Bowen-Margulis μ_{BM} de masse 1, alors c'est l'unique mesure d'entropie maximale; sinon, il n'existe pas de mesure d'entropie maximale.*

Ici, δ_Γ est l'exposant critique du groupe Γ, qui est étroitement lié aux mesures de Patterson-Sullivan:

$$\delta_\Gamma = \limsup_{R\to+\infty} \frac{1}{R} \log N_\Gamma(o,R),$$

où $N_\Gamma(o,R)$ est le nombre de point de l'orbite $\Gamma.o$ du point $o \in X$ sous Γ dans la boule métrique de centre o et rayon R dans X. L'égalité $h_{top} = \delta_\Gamma$ était déjà connu par Manning [51] dans le cas des quotients compacts.

Dans le chapitre 5, je prouve la version suivante de ce théorème pour les quotients de géométries de Hilbert :

Théorème 2 (Section 5.1)**.** *Soit $M = \Omega/\Gamma$ une variété quotient d'un ouvert Ω strictement convexe à bord C^1. Supposons qu'il existe une mesure de Bowen-Margulis μ_{BM} sur HM qui soit de probabilité. Si le flot géodésique n'a pas d'exposant de Lyapunov nul sur l'ensemble non errant, alors la mesure μ_{BM} est l'unique mesure d'entropie maximale et*

$$h_{top} = h_{\mu_{BM}} = \delta_\Gamma.$$

La preuve de ce résultat s'inspire de celle de [61], qui est elle-même basée sur des techniques développées dans les années 70-80 dans l'étude des systèmes non uniformément hyperboliques; l'article déjà mentionné [49] de Ledrappier et Young est l'une des illustrations les plus parlantes de ces techniques. En les adaptant à notre contexte, l'inégalité de Ruelle et son cas d'égalité apparaissent alors comme des conséquences directes, au moins dans le cas des géométries de Hilbert Gromov-hyperboliques:

Théorème 3 (Section 5.2)**.** *Soit (Ω, d_Ω) une géométrie de Hilbert Gromov-hyperbolique et $M = \Omega/\Gamma$ une variété quotient. Pour toute mesure de probabilité φ^t-invariante m, on a*

$$h_\mu \leqslant \int \chi^+ \, dm,$$

avec égalité si et seulement si m a ses mesures conditionnelles instables absolument continues.

(Voir le corps du texte pour une localisation de l'erreur.)

Il est temps de faire une pause pour expliquer le point de vue adopté pour prouver le théorème 1 et étendre les techniques dont j'ai parlé avant. Il s'agit du point de vue développé par Patrick Foulon [33] pour étudier les équations différentielles du second ordre. Les flots géodésiques des métriques de Finsler classiques, qui sont régulières, sont des cas importants dans lesquels le formalisme dynamique de Foulon peut être utilisé. Dans la section 2.1, j'étends ce formalisme au contexte des géométries de Hilbert définies par un ouvert (strictement) convexe à bord de classe C^1; c'est essentiellement le fait que ces géométries soient plates qui permet ici de s'en sortir, malgré le manque de régularité des métriques considérées. En particulier, cela permet de définir un transport parallèle le long des géodésiques qui s'avère contenir toute l'information concernant la dynamique le long de cette géodésique. Par exemple, la propriété d'Anosov du flot géodésique sur un quotient compact, prouvée par Benoist, peut être comprise en termes de transport parallèle.

La remarque cruciale, et un peu déroutante, est que ce transport parallèle n'est en général pas une isométrie. Les différences de comportement du flot géodésique sont essentiellement contenues dans cette observation. En particulier, la somme χ^+ des exposants de Lyapunov positifs peut être exprimée (le long d'une orbite régulière) sous la forme

$$\chi^+ = (n-1) + \eta,$$

formule dans laquelle

$$\eta = \lim_{t \to +\infty} \frac{1}{t} \log |\det T^t|$$

mesure l'effet du transport parallèle T^t sur les volumes. Le théorème 1 est alors une conséquence facile de cette égalité et de la formule (4): on obtient

$$h_{top} \leqslant (n-1) + \int_{HM} \eta \, d\mu_{BM},$$

et $\displaystyle\int_{HM} \eta \, d\mu_{BM} = 0$ pour de simples raisons de symétrie (voir la preuve de la proposition 5.3.1).

Alors que je travaillais sur la preuve du théorème 1, j'avais remarqué qu'on pouvait lire les exposants de Lyapunov d'une géodésique donnée sur la forme du bord $\partial\Omega$ de Ω au point extrémal de la géodésique (voir la proposition 5.4 de [25]). Le chapitre 3 généralise cette remarque à toute géométrie de Hilbert définie par un ouvert strictement convexe à bord de classe C^1. On relie ainsi les exposants de Lyapunov, le transport parallèle et la forme du bord $\partial\Omega$. Comme conséquence de tout cela, j'explique dans la section 3.5 comment les variétés de Lyapunov, tangentes aux sous-espaces apparaissant dans la décomposition de Lyapunov-Osedelets, peuvent être facilement construites. Encore une fois, le fait que la géométrie soit plate est essentiel dans cette construction, et je ne sais donc pas si une telle approche pourrait être envisagée dans le cas des variétés riemanniennes de courbure négative, ou pour des flots d'Anosov plus généraux.

Toutes ces questions n'auraient que peu de sens s'il n'existait pas de quotients pour lesquels se les poser. Une autre partie de mon travail était donc de chercher de tels quotients, en particulier des quotients auxquels le théorème 2 pourrait être appliqué.

Les seuls exemples alors connus étaient les surfaces de volume fini étudiées par Ludovic Marquis dans [57]. Pour ce qui m'intéressait, c'était la décomposition d'une telle surface en une partie compacte et un nombre fini de cusps, dont la géométrie était bien comprise, qui était cruciale. En fait, il est facile de déduire des résultats de [57] que la métrique de Hilbert dans un cusp est bi-Lipschitz équivalente à une métrique riemannienne hyperbolique. Cette simple observation suffit à prouver que le flot géodésique est uniformément hyperbolique, donc sans exposant de Lyapunov nul, et permet d'adapter des approches utilisées en géométrie hyperbolique pour prouver que la mesure de Bowen-Margulis est finie. La preuve du théorème 1 s'applique alors sans modification à cette situation: l'égalité $\displaystyle\int \eta \, d_{\mu_{BM}} = 0$ est une conséquence de la symétrie de la mesure de Bowen-Margulis, qui est un fait très général; quant au cas d'égalité, l'argument donné par Benoist dans [7] prouve qu'il n'y a pas de mesure invariante absolument continue, sauf si la structure est hyperbolique. On obtient ainsi le

Théorème 4 (Théorème 5.3.6). *Soit $M = \Omega/\Gamma$ une surface de volume fini. Alors*

$$h_{top} \leqslant 1,$$

avec égalité si et seulement si M est riemannienne hyperbolique.

(On ne peut plus obtenir ce résultat qui dépend du théorème précédent.)

Les arguments ci-dessous me convinrent que la propriété essentielle était la décomposition de la variété en une partie compacte et une partie "maîtrisable", qui permette d'utiliser les méthodes connues en géométrie hyperbolique. Ce sont de tels quotients qu'il fallait donc rechercher, et ce que nous avons commencé à faire avec Ludovic Marquis.

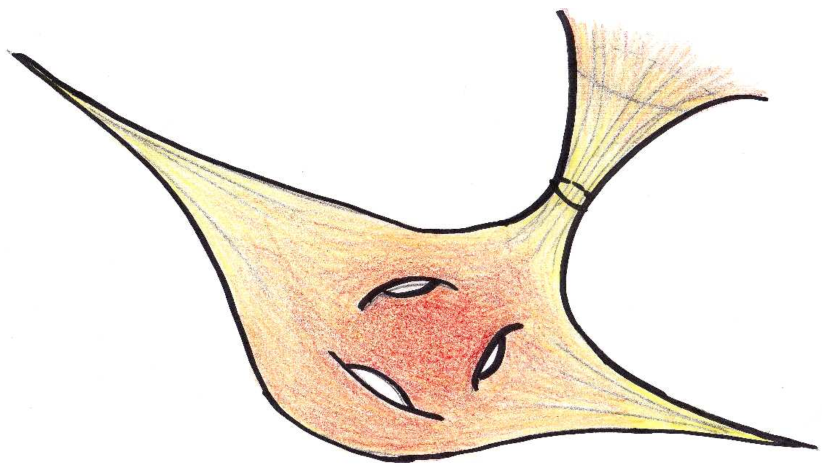

Figure 4: Une surface géométriquement finie

En géométrie hyperbolique, il existe une extension naturelle des variétés de volume fini: les variétés géométriquement finies. Dans ces variétés, le cœur convexe, support de l'ensemble non errant du flot géodésique, est de volume fini. Nous avons donc essayé avec Ludovic Marquis [26] de comprendre cette notion de finitude géométrique en géométrie de Hilbert.
Remarquons que cette notion, qui est aujourd'hui devenue classique, n'était pas vraiment claire avant les travaux de Bowditch [14], qui en a donné diverses définitions équivalentes. Dans le contexte d'une géométrie de Hilbert définie par un ouvert strictement convexe à bord C^1, la définition en termes de points limites est un bon point de départ; c.f. définition 1.4.3. L'étude de tels quotients est encore en cours. Le seul résultat général que nous avons prouvé jusqu'ici est le suivant.

Théorème 5 ([26] et théorème 1.4.8). *Soit $M = \Omega/\Gamma$ une variété géométriquement finie. Alors le cœur convexe de M peut être décomposée en une partie compacte et un nombre fini de cusps.*

(Ce résultat est faux dans cette généralité, comme nous l'avons plus tard remarqué; voir notre article [26]); il est par contre vrai en dimension 2.

Mais cela n'est pas suffisant pour faire marcher la stratégie précédente. En effet, certaines techniques ne fonctionnent plus sans contrôle géométrique des parties cuspidales de la variété. Nous pensions avoir prouvé, à un certain moment, que les cusps avaient essentiellement la même géométrie que ceux des variétés hyperboliques, mais il y avait une erreur importante dans notre approche. Dans cette thèse, je décris ce qu'il se passe en dimension 2, en me basant sur le travail de Marquis [57]. Les principaux résultats concernant le flot géodésique des surfaces géométriquement finies sont donnés dans le

Théorème 6. *Soit $M = \Omega/\Gamma$ une surface géométriquement finie. Alors*

- *le flot géodésique de la métrique de Hilbert est uniformément hyperbolique sur son ensemble non errant (théorème 2.5.2); en particulier, il n'a pas d'exposant de Lyapunov nul;*
- *il existe une mesure de Bowen-Margulis finie (section 4.3).*

Cela montre que les surfaces géométriquement finies satisfont les hypothèses du théorème 2. Les techniques utilisées pour étudier ces surfaces non compactes sont classiques et dépendent uniquement de la bonne compréhension de la géométrie asymptotique des cusps. Par exemple, ces techniques s'appliquent telles quelles aux seuls exemples de variétés de volume fini connus en dimension supérieure, construits par Marquis dans [56].

En fait, ce contrôle de la géométrie des cusps a déjà montré son importance dans l'étude des variétés riemanniennes à courbure négative. L'article [28] en est une bonne illustration: dans celui-ci, Dal'bo, Otal et Peigné parviennent, entre autres choses, à construire des variétés géométriquement finies de courbure négative pincée dont la mesure de Bowen-Margulis est infinie, et pas même ergodique. Dans [29], Dal'bo, Peigné, Picaud et Sambusetti montre que la géométrie asymptotique des cusps a aussi un effet important sur l'entropie volumique. L'entropie volumique h_{vol} d'une variété riemannienne (M, g) mesure la croissance exponentielle du volume des boules métriques dans le revêtement universel $\tilde{M}$:

$$h_{vol} = \limsup_{R \to +\infty} \log vol_g B(o, R),$$

où $B(o, R)$ est la boule métrique de centre arbitraire o et rayon R dans $\tilde{M}$.
Si M est compacte et de courbure négative, Manning [51] a prouvé que entropies volumique et topologique sont égales; sa preuve s'étend sans difficulté aux géométries de Hilbert (proposition 1.6.2). Mais cela devient faux en général pour les variétés de volume fini, et dépend de façon essentielle de la géométrie des cusps:

Théorème (Dal'bo, Peigné, Picaud, Sambusetti [29])**.**

- *Soit M une variété riemannienne à courbure strictement négative, de volume fini. Si M est asymptotiquement $1/4$-pincée, alors $h_{vol} = h_{top}$.*
- *Pour tout $\epsilon > 0$, il existe une variété riemannienne de volume fini et de courbure strictement négative $(1/4 + \epsilon)$-pincée telle que $h_{top} < h_{vol}$.*

En géométrie de Hilbert, je pense que de tels contre-exemples n'existent pas. Là encore, cela dépend de notre compréhension des cusps. En tout cas, pour les surfaces, rien de tel ne peut arriver:

Théorème 7 (Section 4.4)**.** *Soit $M = \Omega/\Gamma$ une surface de volume fini. Alors*

$$h_{vol} = h_{top} = \delta_\Gamma.$$

Tout cela admet les corollaires suivant concernant l'entropie volumique de certaines géométries de Hilbert:

Corollaire 10 (Corollaire 5.3.5)**.** *Soit $\Omega \subset \mathbb{RP}^n$ un ouvert proprement convexe et strictement convexe qui admet un quotient compact. Alors son entropie volumique h_{vol} satisfait à l'inégalité*

$$h_{vol} \leqslant n - 1,$$

avec égalité si et seulement si Ω est un ellipsoïde.

Corollaire 11 (Corollaire 5.3.7)**.** *Soit $\Omega \subset \mathbb{RP}^2$ un ouvert proprement convexe qui admet un quotient de volume fini. Alors son entropie volumique h_{vol} satisfait à l'inégalité*

$$h_{vol} \leqslant 1,$$

avec égalité si et seulement si Ω est une ellipse.

On conjecture que l'entropie volumique d'une géométrie de Hilbert de dimension n est toujours inférieure à $n-1$. Cette conjecture a été prouvée en dimension 2 par Berck, Bernig et Vernicos [10], qui ont aussi prouvé l'égalité $h_{vol} = n-1$ pour un convexe dont le bord est de classe $C^{1,1}$. Les deux derniers corollaires confirment cette conjecture pour une certaine classe de géométries de Hilbert, et fournissent aussi une infinité d'exemples pour lesquels l'entropie volumique est strictement comprise entre 0 et $n-1$.

Finissons cette introduction par une description rapide de ce que l'on trouvera dans les différents chapitres de cette thèse.

La chapitre 1 fait d'abord une courte introduction aux géométries de Hilbert et rappelle des notions et résultats déjà connus. Les quotients des géométries de Hilbert sont étudiés dans les sections 1.3 et 1.4. On trouve là certains des nouveaux résultats géométriques de [26]: la section 1.3 décrit les groupes paraboliques et la géométrie des cusps; la section 1.4.3 introduit la notion de quotient géométriquement fini et leur cœur convexe est décomposé par le théorème 1.4.8; le cas des surfaces est plus précisément considéré dans la section 1.4.4.

Le chapitre 2 commence l'étude du flot géodésique des métriques de Hilbert. On étend d'abord le formalisme dynamique de Foulon et on montre son utilité en géométrie de Hilbert: les résultats fondamentaux sont les propositions 2.4.1 and 2.4.5. La section 2.5 termine ce chapitre en prouvant l'uniforme hyperbolicité du flot géodésique sur les quotients compacts et les surfaces géométriquement finies.

Dans le chapitre 3, on s'intéresse aux exposants de Lyapunov du flot géodésique. On montre en particulier que le théorème d'Oseledets peut être appliqué à toute variété quotient. Dans la section 3.4, on explique les liens entre transport parallèle, exposant de Lyapunov et la forme du bord à l'infini. Pour cela, on a besoin d'introduire une nouvelle propriété de régularité des fonctions convexes, qu'en particulier on prouve être projectivement invariante, et donc adaptée à notre problème. Comme conséquence, on explique dans la section 3.5 comment on peut facilement définir les variétés de Lyapunov.

Le chapitre 4 étudie les propriétés des mesures de Patterson-Sullivan et de Bowen-Margulis. On explique d'abord pourquoi certains théorèmes connus pour les variétés riemanniennes de courbure négative, entre autres le théorème 4.2.4, restent valables dans notre contexte. La section 4.3 prouve que toute mesure de Bowen-Margulis d'une surface géométriquement finie est finie. Dans la dernière section, on montre qu'exposant critique et entropie volumique coïncident pour une surface de volume fini.

Dans le dernier chapitre, on rappelle d'abord comment construire des partitions mesurables qui permettent de calculer efficacement des entropies, et on applique ces techniques pour obtenir le théorème 2. L'inégalité de Ruelle et son cas d'égalité sont alors étendues à certains quotients non compacts. Comme conséquence, on obtient les théorèmes 1 et 4 ainsi que leurs équivalents volumiques.

Chapter 1

Hilbert geometries and its quotients

This chapter consists of preliminaries. We define Hilbert geometries, describe some of its general properties, as well as some tools we will use all along the text. We study isometries of Hilbert geometries. We describe compact quotients, introduce the notion of geometrically finite manifolds, and give a complete presentation of the 2-dimensional case. We end this chapter by introducing the concepts of topological and volume entropies.

1.1 General metric properties

1.1.1 Definition

Take the open unit ball B in the Euclidean space $(\mathbb{R}^n, |\ .\ |)$, and define a metric on B by setting

$$d_B(x, y) = \frac{1}{2} \log[a, b, x, y],$$

for any two distinct points $x,\ y \in B$, a and b being the two intersection points of the line (xy) and the boundary ∂B of B (see figure 1.1); $[a, b, x, y]$ denotes the cross-ratio of the four points:

$$[a, b, x, y] = \frac{|ax|/|bx|}{|ay|/|by|}.$$

(B, d_B) is the Beltrami model of the hyperbolic space $\mathbb{H}^n$. In this model, the geodesics are the lines.

At the end of the nineteenth century, Hilbert [40] generalized this construction by replacing the unit ball B by any bounded convex subset Ω of $\mathbb{R}^n$, the distance being given by the same formula:

$$d_\Omega(x, y) = \frac{1}{2} \log[a, b, x, y].$$

It leads to a well-defined complete metric space (Ω, d_Ω), and the topology induced by the metric is the same as the one induced by $\mathbb{R}^n$ on Ω (See [3]). Hilbert's main remark was that lines are still

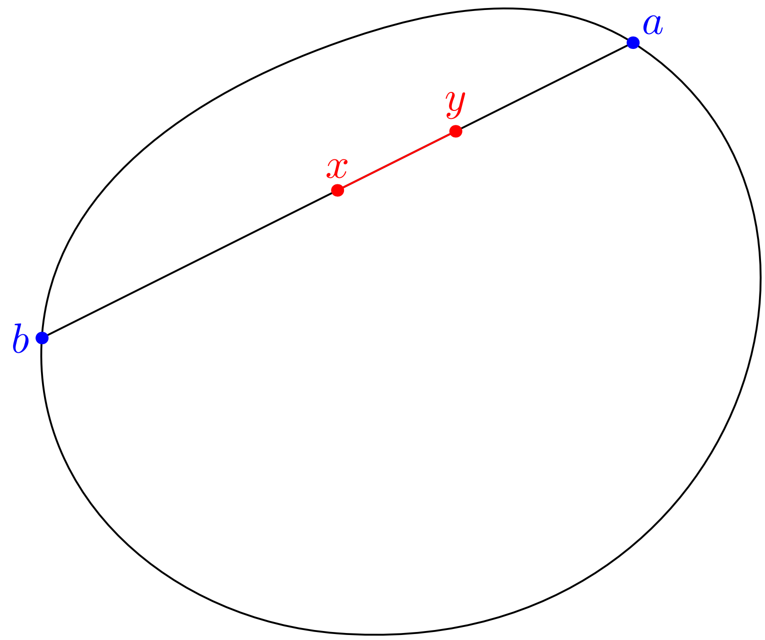

Figure 1.1: The Hilbert distance

geodesics, that is, the length of the segment $[xy]$ is equal to the distance $d_\Omega(x,y)$. The length of a curve $c : [0,1] \longrightarrow \Omega$ is here defined as

$$\sup\left\{\sum_{i=0}^{n-1} d_\Omega(c(t_i), c(t_{i+1}))\right\},$$

where the supremum is taken over all finite partitions $0 = t_0 < t_1 < \cdots < t_n = 1$ of $[0,1]$. In particular, that implies that (Ω, d_Ω) is a geodesic space. Such a space (Ω, d_Ω) will be called a **Hilbert geometry**.

Historically, these spaces are examples that Hilbert gave for his fourth problem [41]:

> We are asking, then, for a geometry in which all the axioms of ordinary Euclidean geometry hold, and in particular all the congruence axioms except the one of the congruence of triangles (or all except the theorem of the equality of the base angles in the isosceles triangle), and in which, besides, the proposition that in every triangle the sum of two sides is greater than the third is assumed as a particular axiom.

Stated in this form, the problem was too vague to say it has been solved so far. For more details about this, we refer to [1].

Consider a bounded open convex set Ω of $\mathbb{R}^n$, and a projective transformation $g \in PGL(n+1,\mathbb{R})$ such that $g\Omega$ is still bounded. Since cross-ratios are preserved by projective transformations, the space $(g\Omega, d_{g\Omega})$ is obviously isometric to (Ω, d_Ω). Also we see that a projective transformation preserving Ω is an isometry of (Ω, d_Ω). Thus, it seems more coherent to see Ω as a subset of the projective space $\mathbb{RP}^n$ and not of $\mathbb{R}^n$. For example, the Beltrami model of $\mathbb{H}^n$ is defined more generally on an ellipsoid, which is projectively equivalent to the unit ball in $\mathbb{R}^n$. In all this text, an ellipsoid has to be understood as the hyperbolic space, and conversely...

We will say that a subset Ω of $\mathbb{RP}^n$ is **convex** if the intersection of Ω with any projective line in $\mathbb{RP}^n$ is connected. A convex subset Ω of $\mathbb{RP}^n$ is **proper** if there exists a projective hyperspace that does not intersect Ω; equivalently, Ω is proper if there exists an affine chart in which Ω appears as a relatively compact set.

Let $p : \mathbb{R}^{n+1} \smallsetminus \{0\} \longrightarrow \mathbb{RP}^n$ be the natural projection. If Ω is a convex proper open subset of $\mathbb{RP}^n$, then $p^{-1}(\Omega)$ consists of two disjoint open cones. It is sometimes useful to think of Ω as one of these cones.

The Hilbert distance d_Ω on a convex proper open subset $\Omega \subset \mathbb{RP}^n$ is defined by considering any affine chart that makes Ω appear as a relatively compact subset of $\mathbb{R}^n$. We can also define it directly on one of the cones of $p^{-1}(\Omega)$.

We will say that a proper convex set Ω is strictly convex if there is an affine chart in which it appears as a relatively compact strictly convex set.

It is often clever to look at the dual geometry defined by the dual convex set Ω^*. If C is one of the cones of $p^{-1}(\Omega)$ in $\mathbb{R}^{n+1}$, the dual convex cone C^* in $(\mathbb{R}^{n+1})^*$ is

$$C^* = \{f \in (\mathbb{R}^{n+1})^*,\ \forall x \in C,\ f(x) > 0\},$$

and $\Omega^* = p(C^*)$ is its trace. Of course, duality is an involution: $(\Omega^*)^* = \Omega$.

The boundary of Ω^* consists of those linear forms whose kernel is an hyperplane tangent to the boundary $\partial\Omega$ of Ω. We will often think of $\partial\Omega^*$ as the set of spaces tangent to $\partial\Omega$. If $\partial\Omega$ is not C^1 at some point x (x is a "corner"), then there are several tangent spaces to $\partial\Omega$ at x, and this "creates" a flat part in $\partial\Omega^*$; and conversely. Intuitively, duality transforms corners into flats. In particular, $\partial\Omega$ is C^1 if and only if Ω^* is strictly convex. When Ω is strictly convex with C^1 boundary, there is then a natural identification between the boundaries $\partial\Omega$ and $\partial\Omega^*$.

1.1.2 The Finsler metric

Among all Hilbert geometries, defined by different convex sets, only the one defined by an ellipsoid is Riemannian, that is, there is a Riemannian metric which generates the Hilbert metric. In all the other cases, the metric is not Riemannian but is still Finslerian. That means that, instead of having a scalar product on each tangent space $T_x\Omega$, we have a norm $F(x, .)$.

Take a convex proper open subset $\Omega \subset \mathbb{RP}^n$, that we see as a bounded convex set in an affine chart $\mathbb{R}^n$ equipped with any Euclidean metric $|\ .\ |$. For $x \in \Omega$, the Finsler norm on $T_x\Omega$ is defined for $\xi \in T_x\Omega$ by

$$F(x,\xi) = \frac{|\xi|}{2}\left(\frac{1}{|xx^+|} + \frac{1}{|xx^-|}\right), \tag{1.1}$$

where $x^+,\ x^-$ are the intersections of the line $\{x + \lambda\xi\}_{\lambda\in\mathbb{R}}$ with the boundary $\partial\Omega$ (see figure 1.2).

The Hilbert length of a C^1 curve $c : [0,1] \longrightarrow \Omega$ can now be computed as the integral

$$l(c) = \int_0^1 F(\dot{c}(t))\ dt,$$

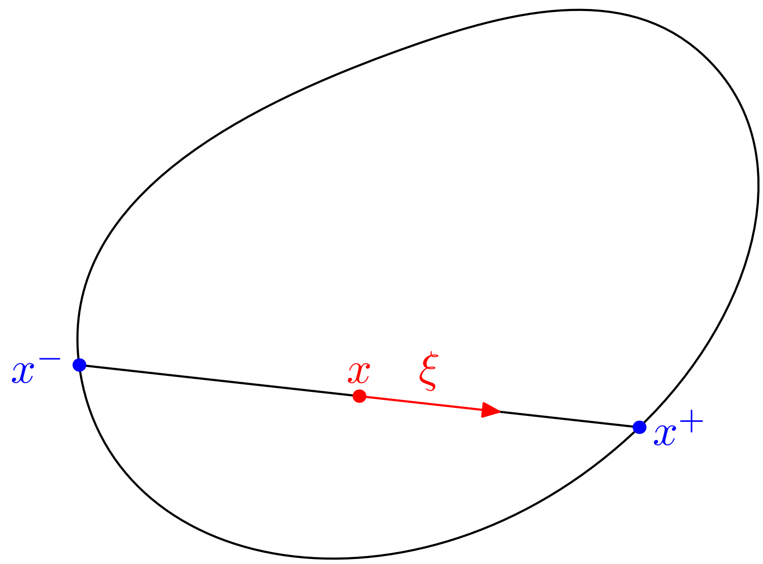

Figure 1.2: The Finsler metric

and the distance d_Ω is induced by the Finsler norm in the sense that

$$d_\Omega(x, y) = \inf l(c),$$

where the infimum is taken with respect to all C^1 curves from x to y, that is, $c(0) = x$, $c(1) = y$.

We say that the Hilbert metric is of class C^p, $p \in \mathbb{N}$, if $F : T\Omega \setminus \{0\} \longrightarrow \mathbb{R}$ is a C^p function. From the formula (1.1), we see that the Hilbert metric has indeed at least the same regularity as the boundary $\partial\Omega$.

1.1.3 Intuitive considerations and restrictions

Consider the Hilbert geometry defined by a convex proper open subset $\Omega \subset \mathbb{RP}^n$.

If lines are always geodesics, there might be geodesics which are not lines, as illustrated by figure 1.3. On this figure, the path in blue and the path in red[1] are geodesics: projections are homographies, hence

$$d_\Omega(x, z) = \frac{1}{2}\log[a, b, x, z] = \frac{1}{2}\log[a', b', x, z'] = d_\Omega(x, z'),$$

and similarly, $d_\Omega(z, y) = d_\Omega(z', y)$.

The situation on this figure is essentially the only one where there can be other geodesics. In particular, that does not occur if Ω is strictly convex: the Hilbert geometry defined by a strictly

[1] if printed in color...

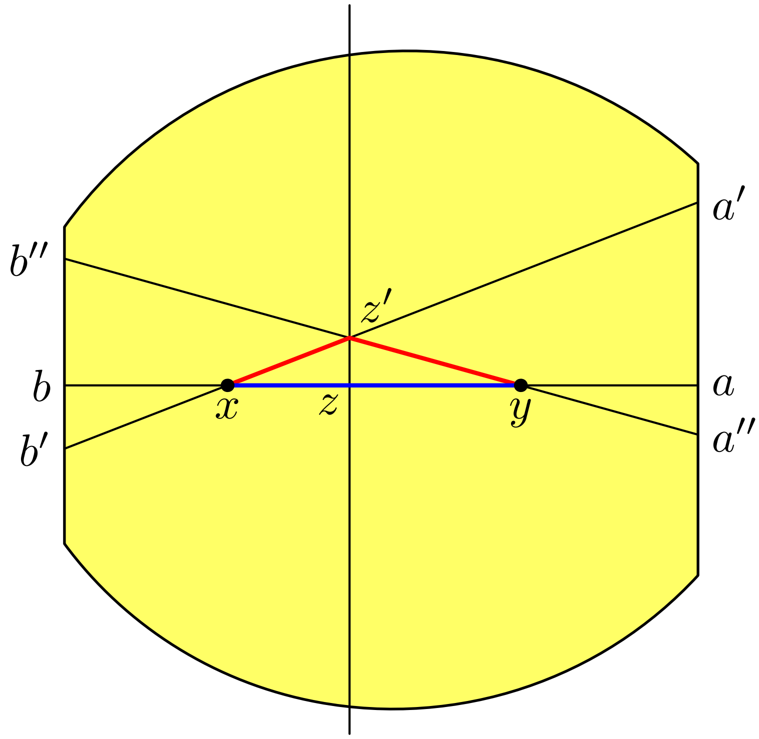

Figure 1.3: A geodesic which is not a line

convex set is uniquely geodesic.

It is important here to say that geodesics are defined in metric terms: a continuous curve joining x to y is a geodesic segment if its length is equal to $d_\Omega(x, y)$. In particular, there are no geodesics equations involved in this context, that would arise from a variational problem. Anyway, that would not make sense in the case $\partial\Omega$ is not C^2.

Nevertheless, if $\partial\Omega$ is C^2 *with definite positive Hessian*, then we get geodesic equations as usual, and the solutions are the lines. This assumption is the one which usually appears in the general definition of a Finsler metric; but as proved by Socié-Méthou [71], such an assumption is too much restrictive if we want to consider quotient manifolds modeled on Hilbert geometries:

Theorem 1.1.1 ([71]). *Let $\Omega \subset \mathbb{RP}^n$ be a convex proper open set. Assume the boundary $\partial\Omega$ is C^2 with definite positive Hessian. Then the group of isometries $Isom(\Omega, d_\Omega)$ of (Ω, d_Ω) is compact, unless Ω is an ellipsoid.*

There are lots of differences between the strictly convex and non strictly convex cases, or between convex sets with C^1 boundary or not, especially about asymptotic geometry. I will try to give an intuitive feeling about these differences in the 2-dimensional case.

About strict convexity

Consider a bounded open convex set $\Omega \subset \mathbb{R}^2$, and pick two distinct points p and q in $\partial\Omega$, which are not contained in a segment of $\partial\Omega$. Then, for any two sequences of points (p_n) and (q_n) in Ω converging to p and q in $\overline{\Omega} = \Omega \cup \partial\Omega \subset \mathbb{R}^n$, the distance $d_\Omega(p_n, q_n)$ tends to $+\infty$ when $n \to +\infty$. Assume now that p and q are are contained in a segment $[ab]$ in $\partial\Omega$, which we assume is maximal, that is, it is not contained in a larger segment of $\partial\Omega$. Consider two lines $c_p, c_q : [0, +\infty) \longrightarrow \Omega$ of Ω ending at p and q, that is,

$$\lim_{t\to+\infty} c_p(t) = p, \ \lim_{t\to+\infty} c_q(t) = q,$$

in $\overline{\Omega}$. These two geodesics are asymptotic: the function $t \longmapsto d_\Omega(\gamma_p(t), \gamma_q(t))$ is bounded. For example, in figure 1.4, we can parametrize the geodesics $c_p(t)$ and $c_q(t)$, that is, we can choose $c_p(0)$ and $c_q(0)$, in such a way that

$$\lim_{t\to+\infty} d_\Omega(c_p(t), c_q(t)) = \frac{1}{2}\log[a, b, p, q].$$

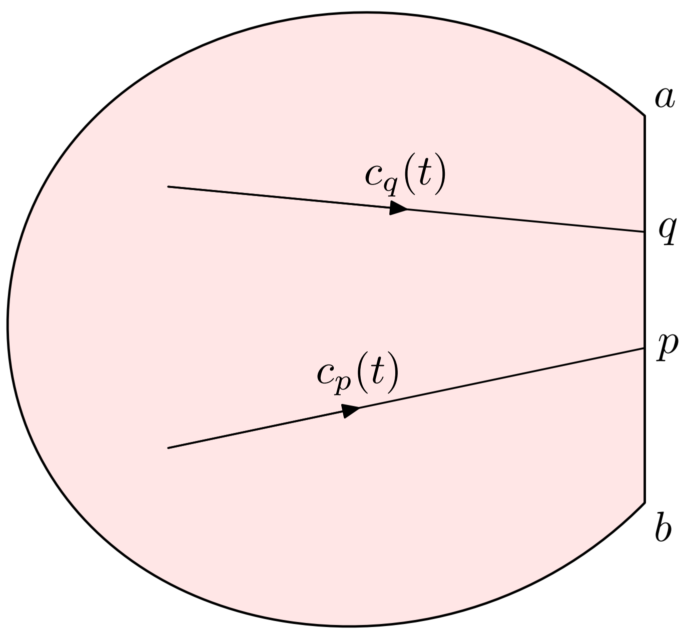

Figure 1.4: Asymptotic geodesics that do not converge to the same point

This can be stated more precisely in the following way. If p is contained in the maximal nontrivial segment $[ab]$ of $\partial\Omega$, and (p_k) is any sequence in Ω converging to p in $\overline{\Omega}$, then the closed Hilbert ball of radius 1 centered at p_k converges in $\mathbb{R}^n$ to the segment $[qq'] \subset [ab]$, where q and q' are the points of $[ab]$ such that

$$\frac{1}{2}\log[a, b, p, q] = \frac{1}{2}\log[a, b, q', p] = 1.$$

On the contrary, if p is not contained in a nontrivial segment of $\partial\Omega$, then the same sequence of balls converges in $\overline{\Omega}$ to the point p.
That means that the boundary at infinity defined by equivalence classes of asymptotic geodesics is not given by $\partial\Omega$ when Ω is not strictly convex.

About C^1 regularity of the boundary

Another problem occurs at a point where the boundary of Ω is not C^1. Take for example the vertex p of a triangle Ω, and consider two distinct lines γ and γ' ending at p. Then the distance $d(\gamma(t), \gamma'(t))$ does not tend to 0 when t goes to $+\infty$. The same works at a non-C^1 point of the boundary of any convex set.
This does not occur if $\partial\Omega$ is C^1 at p: for two lines $\gamma, \gamma' : \mathbb{R} \longrightarrow \Omega$ ending at p, there exists a time $t_0 \in \mathbb{R}$ such that

$$\lim_{t\to+\infty} d_\Omega(\gamma(t), \gamma'(t+t_0)) = 0. \tag{1.2}$$

Indeed, one has to choose $t_0 = \pm \lim_{t\to+\infty} d_\Omega(\gamma(t), \gamma'(t))$, with the appropriate sign.
This property (1.2) is of crucial use when working on the universal cover of a manifold of pinched negative curvature, and fails when the curvature is allowed to be zero.

In this work, we are interested in those Hilbert geometries which exhibit some hyperbolic behaviour, and more especially in what regards the geodesic flow. The last remarks explain why we restrict ourselves to the geometries which are defined by a strictly convex set with C^1 boundary. Another reason is the following: all the tools that we will use require the C^1-regularity...

Let us emphasize that strict convexity and C^1-regularity tend to appear by pair when we consider quotient manifolds. For example, theorem 1.4.2 tells us that if Ω admits a compact quotient, then either Ω is strictly convex with C^1 boundary, or it is not strictly convex and the boundary is not C^1. This can be seen as a consequence of duality: if Ω admits a compact quotient by a group of projective transformations, then its dual Ω^* also. A similar result is expected for geometrically finite quotients; that is one of the goals of an article I am working on with Ludovic Marquis [26]. In fact, this has been proved by Cooper, Long and Tillmann [30] for finit volume quotients. For geometrically finit quotients, it is still not clear what we can expect.

1.1.4 Global results about Hilbert geometries

We review here some results about the global properties of Hilbert geometries. For more insights about it, have a look at the very clear and complete exposition in [71].

What is globally expected is that Hilbert geometries are geometries in between Euclidean and hyperbolic ones. As already remarked in the preceding section, a hyperbolic behaviour implies strict convexity and C^1-regularity of the boundary.
It is important to remark that Hilbert geometries cannot be classified by their local behaviour: for example, a Hilbert geometry is not $\mathrm{CAT}(k)$ for any $k \in \mathbb{R}$, except in the case of the ellipsoid. Large scale properties are more appropriate.

The two following results are a good example of what can be said.

Theorem 1.1.2.
- *[22] If $\partial\Omega$ is C^2 with definite positive Hessian, then the metric space (Ω, d_Ω) is bi-Lipschitz equivalent to the hyperbolic space $\mathbb{H}^n$.*
- *[23] [11] [73] (Ω, d_Ω) is bi-Lipschitz equivalent to the Euclidean space if and only if Ω is a convex polytope, that is, the convex hull of a finite number of points.*

The case where $\partial\Omega$ is C^2 with definite positive Hessian is exactly this case where the Finsler geometry is of a classical type, with strongly convex unit balls.

Except for the case of polytopes and without further assumptions, not a lot is known about the geometries defined by convex sets whose boundary is not C^2. Nevertheless, Yves Benoist gave a beautiful characterization of Hilbert geometries that are Gromov-hyperbolic:

Theorem 1.1.3 ([6]). *A Hilbert geometry (Ω, d_Ω) is Gromov-hyperbolic if and only if Ω is quasi-symmetrically convex.*

The notion of quasi-symmetric convexity was introduced by Benoist in the same paper. It is not essential here, so we refer to his article for more details. Just notice the significant fact that quasi-symmetric convexity implies strict convexity and $C^{1+\epsilon}$-regularity of the boundary, for some $\epsilon > 0$.

Let us recall instead the definition of Gromov-hyperbolic spaces. Let (X, d) be a metric space, and fix an arbitrary point of reference $o \in X$. The Gromov-product based at o of two points x and y in X is defined as

$$(x|y)_o = \frac{1}{2}(d(x,o) + d(o,y) - d(x,y)).$$

The space (X, d) is then said to be **Gromov-hyperbolic** if there exists some $\delta \geqslant 0$ such that for any $x, y, z \in X$,

$$(x|z)_o \geqslant \min\{(x|y)_o, (y|z)_o)\} - \delta.$$

The space is also said to be **δ-hyperbolic**.

A more intuitive definition can be given for proper[2] geodesic metric spaces (see figure 1.5): (X, d) is Gromov-hyperbolic if there is some $\delta > 0$ such that any geodesic triangle $xyz \subset X$ of vertices $x, y, z \in X$ is δ-thin, that is, for any point p on the side $[xz]$,

$$\min\{d(p, [xy]), d(x, [yz])\} \leqslant \delta.$$

Obviously, the hyperbolic space $\mathbb{H}^n$ is a Gromov-hyperbolic space. The extremal case is the one given by trees: equipped with the word metric, trees are indeed 0-hyperbolic, since the triangles have no interior.
Using Cayley graphs, Gromov introduced in [37] the now classical notion of hyperbolic group: a finitely generated group G is Gromov-hyperbolic if its Cayley graph equipped with the word metric is a Gromov-hyperbolic metric space. The property does not depend on the chosen set of generators, but the constant δ of hyperbolicity may depend on it. For example, the fundamental groups of compact surfaces of genus $g \geqslant 2$ are Gromov-hyperbolic. More generally, if a compact manifold carries a metric of negative curvature, then its fundamental group is Gromov-hyperbolic.

[2] A metric space is proper if metric balls are compact.

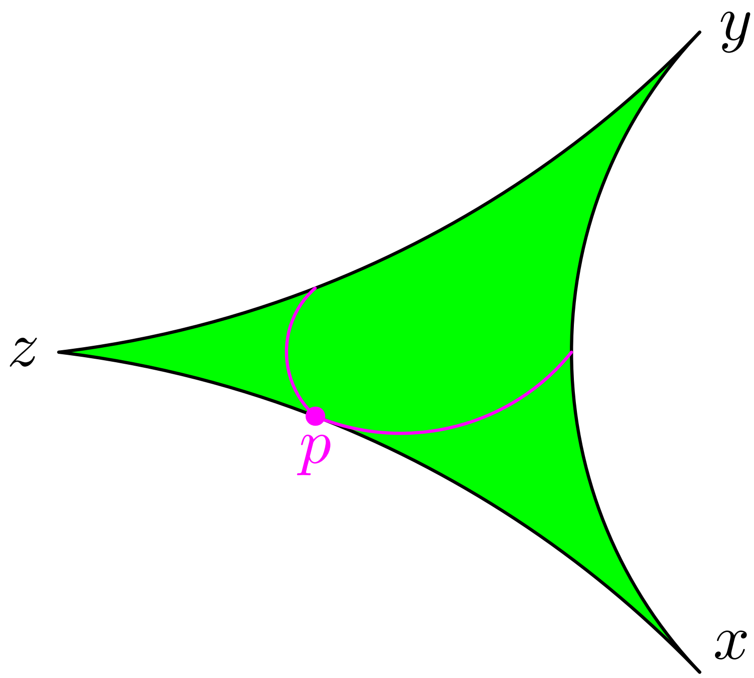

Figure 1.5: A Gromov-hyperbolic triangle

It is important to notice that Gromov-hyperbolicity is not a local property. One just wants the geometry at large scale to be "like in the hyperbolic space". In particular, the notion of Gromov-hyperbolicity is invariant by quasi-isometry: if (X, d) and (X', d') are two metric spaces, a quasi-isometry between X and X' is a map $f : X \longrightarrow X'$ such that

- for any $x, y \in X$,
$$\frac{1}{a}d(x,y) - b \leqslant d'(f(x), f(y)) \leqslant ad(x,y) + b,$$
for some constants $a > 0, b \geqslant 0$;
- there is a constant $c \geq 0$ such that, for any $x' \in X'$, there is some $x \in X$ satisfying $d'(f(x), x') \leqslant c$.

For example, if Γ is a cocompact subgroup of isometries of a metric space (X, d), then (the Cayley graph of) Γ and X are quasi-isometric, and X is Gromov-hyperbolic if and only if Γ is Gromov-hyperbolic.

Gromov-hyperbolicity is the kind of coarse properties that can be expected for Hilbert geometries. In fact, as we will see in the next section, lots of tools that are defined and used in Gromov-hyperbolic spaces can be also considered in the Hilbert geometry defined by a strictly convex set with C^1 boundary.

From now on, unless it is explicitly mentioned, we consider only Hilbert geometries defined by strictly convex proper open sets $\Omega \subset \mathbb{RP}^n$ with C^1 boundary.

1.2 The boundary of Hilbert geometries

Let $\Omega \subset \mathbb{RP}^n$ be a strictly convex proper open set with C^1 boundary. As already remarked, the geometric boundary $\partial\Omega$ corresponds to the geodesic boundary at infinity. We now define some classical tools that are used to study Hadamard manifolds or Gromov-hyperbolic spaces.
Let x and y be in Ω. The shadow of the ball $B(y,r)$ of radius $r \geqslant 0$ about y as seen from x is denoted by $\mathcal{O}_r(x,y)$: it is the subset of $\partial\Omega$ consisting of points ξ such that the geodesic ray $[x\xi)$ intersects $B(y,r)$. The light cone $\mathcal{F}_r(x,y)$ from x and of base $B(y,r)$ is the set of points p in Ω such that the ray $[xp)$ intersects $B(y,r)$; in other words, $\mathcal{F}_r(x,y)$ is the union of all rays $[x\xi)$ for $\xi \in \mathcal{O}_r(x,y)$. See figure 1.6.

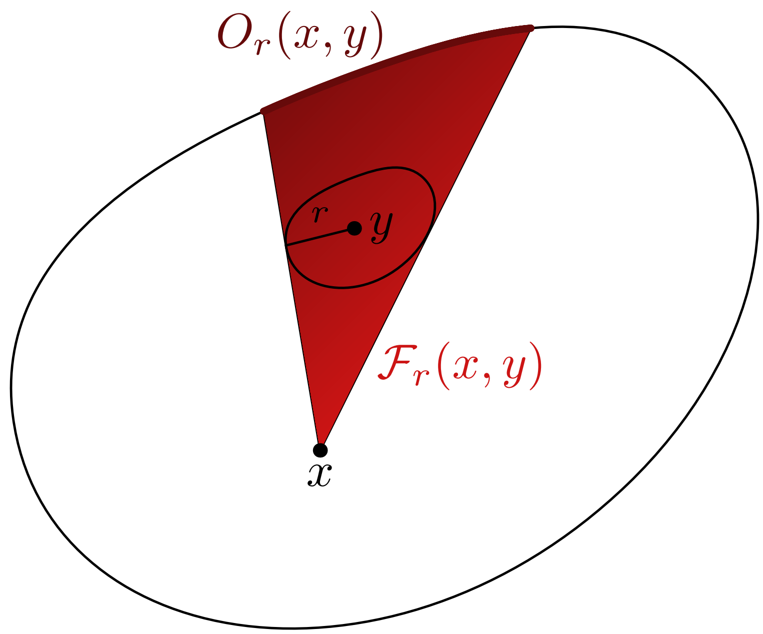

Figure 1.6: Shadows and lightcones

The **Gromov-product** based at o of two points x and y in Ω was already defined as

$$(x|y)_o = \frac{1}{2}(d_\Omega(x,o) + d_\Omega(o,y) - d_\Omega(x,y)).$$

When Ω is strictly convex with C^1 boundary, the Gromov product can be extended continuously to $\overline{\Omega} \times \overline{\Omega} \smallsetminus \Delta$, where $\Delta = \{(x,x),\ x \in \partial\Omega\}$ is the diagonal: that is lemma 5.2 in [8]. We can anyway extend the Gromov product to the whole of $\overline{\Omega} \times \overline{\Omega}$ by saying that $(x|x)_o = +\infty$ if $x \in \partial\Omega$.

The **Busemann function** based at $\xi \in \partial\Omega$ is defined by

$$b_\xi(x, y) = \lim_{p \to \xi} d_\Omega(x, p) - d_\Omega(y, p) = (\xi|y)_x - (\xi|x)_y,$$

which, in some sense, measures the (signed) distance from x to y in Ω as seen from the point $\xi \in \partial\Omega$. A particular expression for b is given by

$$b_\xi(x, y) = \lim_{t \to +\infty} d_\Omega(x, \gamma(t)) - t,$$

where γ is the geodesic leaving y at $t = 0$ to ξ. When ξ is fixed, then b_ξ is a surjective map from $\Omega \times \Omega$ onto $\mathbb{R}$. When x and y are fixed, then $b_.(x, y) : \partial\Omega \to \mathbb{R}$ is bounded by a constant $C = C(x, y)$.

The following lemma will be used many times in chapter 4:

Lemma 1.2.1. *1. For any $x, y \in \overline{\Omega}$ such that $y \in \mathcal{F}_r(x, o)$, we have $(x|y)_o \leqslant r$.*

2. For any $\xi \in \mathcal{O}_r(x, y)$, we have

$$d_\Omega(x, y) - 2r \leqslant b_\xi(x, y) \leqslant d_\Omega(x, y).$$

Proof. 1. Assume $x, y \in \Omega$. The line (xy) intersects $B(o, r)$ and we can pick z in this intersection. We have

$$d_\Omega(x, o) \leqslant d_\Omega(x, z) + d_\Omega(z, o) \leqslant d_\Omega(x, z) + r$$

and

$$d_\Omega(y, o) \leqslant d_\Omega(y, z) + r,$$

so that $2(x|y)_o = d_\Omega(x, o) + d_\Omega(y, o) - d_\Omega(x, y) \leqslant 2r$. By continuity of the Gromov product, this also holds if $x, y \in \partial\Omega$.

2. From the triangular inequality, we have $|b_\xi(x, y)| \leqslant d_\Omega(x, y)$, hence the upper bound. For the lower one, let z be any point in $B(y, r) \cap [x\xi]$, and $[x\xi) : [0, +\infty) \longrightarrow \Omega$ be the geodesic ray from x to ξ. We have

$$\begin{aligned} b_\xi(x, y) &= \lim_{t \to +\infty} d_\Omega(x, [x\xi)(t)) - d_\Omega(y, [x\xi)(t)) \\ &= d_\Omega(x, z) + \lim_{t \to +\infty} d_\Omega(z, [x\xi)(t)) - d_\Omega(y, [x\xi)(t)) \\ &= d_\Omega(x, z) + b_\xi(z, y). \end{aligned}$$

But since $z \in B(y, r)$, $d_\Omega(x, z) \geqslant d_\Omega(x, y) - r$ and $|b_\xi(z, y)| \leqslant r$, hence the result.

□

The **horosphere** passing through $x \in \Omega$ and based at $\xi \in \partial\Omega$ is the set

$$\mathcal{H}_\xi(x) = \{y \in \Omega,\ b_\xi(x, y) = 0\}.$$

$\mathcal{H}_\xi(x)$ is also the limit when p tends to ξ of the metric spheres $B(p, d_\Omega(p,x))$ about p passing through x. In some sense, the points on $\mathcal{H}_\xi(x)$ are those which are as far from ξ as x is.
The (open) horoball $H_\xi(x)$ defined by $x \in \Omega$ and based at $\xi \in \partial\Omega$ is the "interior" of the horosphere $\mathcal{H}_\xi(x)$, that is, the set

$$H_\xi(x) = \{y \in \Omega,\ b_\xi(x,y) > 0\}.$$

It is easy to see that horospheres have the same kind of regularity as the boundary of Ω.

1.3 Isometries of Hilbert geometries

1.3.1 The group of isometries of a Hilbert geometry

Let (Ω, d_Ω) be *any* Hilbert geometry. Its group of isometries $Isom(\Omega, d_\Omega)$ contains the subgroup consisting of projective transformations preserving Ω:

$$Aut(\Omega) = \{g \in PGL(n+1, \mathbb{R}),\ g(\Omega) = \Omega\}.$$

If Ω is strictly convex, all the geodesics are lines and this implies, as remarked by Pierre de la Harpe in [31], that $Aut(\Omega) = Isom(\Omega, d_\Omega)$. In the same paper, de la Harpe constructed the essentially unique nonprojective isometry of the triangle: he proved that if Ω is a triangle, then $Aut(\Omega)$ has index 2 in $Isom(\Omega, d_\Omega)$. In general, it is not known when the two groups coincide.

What follows now is an important part of the article in preparation [26], where the notion of geometrically finite quotients of Hilbert geometries is investigated. We omit some of the proofs and only indicate the results that we will use in the rest of the text. More will appear in [26].

1.3.2 Classification of isometries

Let (Ω, d_Ω) be *any* Hilbert geometry. For $g \in Isom(\Omega, d_\Omega)$, we denote by

$$\tau(g) = \inf_{x\in\Omega} d_\Omega(x, gx),$$

the displacement of g and we say that g is

- **elliptic** if $\tau(g) = 0$ and the infimum is attained, i.e. g fixes a point in Ω;
- **parabolic** if $\tau(g) = 0$ and the infimum is not attained;
- **hyperbolic** if $\tau(g) > 0$ and the infimum is attained;
- **quasi-hyperbolic** if $\tau(g) > 0$ and the infimum is not attained.

As in the hyperbolic space, there are no quasi-hyperbolic isometries if Ω is strictly convex. The more precise result of the following theorem can be seen as a consequence of the intuitive considerations that we made in section 1.1.3.

Theorem 1.3.1 ([26]). *Let $\Omega \subset \mathbb{RP}^n$ be a strictly convex proper open set with C^1 boundary. An isometry g of (Ω, d_Ω) is of one of the following types:*

- *g is elliptic;*
- *g is parabolic; g fixes exactly one point $p \in \partial\Omega$ and for any $x \in \overline{\Omega}$,*
$$\lim_{n\to\pm\infty} g^n x = p;$$
- *g is hyperbolic: it fixes exactly two points g^+ and g^- in $\partial\Omega$ and for any $x \in \overline{\Omega}$,*
$$\lim_{n\to\pm\infty} g^n x = g^{\pm}.$$

Let g be a hyperbolic isometry of (Ω, d_Ω). If we see g as an element of $SL(n+1,\mathbb{R})$, then the last theorem says that g is *biproximal*: associated to the stable lines g^+ and g^-, are their two real eigenvalues λ_g^+, which is the largest eigenvalue (in modulus), and λ_g^-, which is the smallest one; these two eigenvalues are isolated: their eigenspaces are exactly the lines g^+ and g^-. g acts on the segment $[g^-g^+] \subset \overline{\Omega}$ as a translation of length

$$\frac{1}{2}\log\frac{\lambda_g^+}{\lambda_g^-} = \tau(g) = d_\Omega(x, gx),$$

for any $x \in (g^-g^+)$.

1.3.3 Parabolic subgroups

A **parabolic subgroup** of isometries is a nontrivial subgroup of $Isom(\Omega, d_\Omega)$ whose elements but the identity are all parabolic isometries which fix the same point at infinity. If Γ is a given subgroup of $Isom(\Omega, d_\Omega)$, a **maximal** parabolic subgroup is a parabolic subgroup containing all the parabolic isometries of Γ fixing a given point.

In hyperbolic geometry, a parabolic subgroup fixing the point p at infinity acts on $\partial\mathbb{H}^n \smallsetminus \{p\}$ by Euclidean transformations, and discrete parabolic subgroups are thus well understood thanks to Bieberbach theorems. In Hilbert geometry, we do not know if it stays true but we hope so (or maybe not). Here are some partial results in this direction.

Lemma 1.3.2 ([26])**.** *Let $\Omega \subset \mathbb{RP}^n$ be a strictly convex proper open set with C^1 boundary, and $g \in Isom(\Omega, d_\Omega)$ a parabolic isometry fixing $p \in \partial\Omega$. Then g preserves every horosphere based at p.*

Proof. Busemann functions based at p are invariant by g: for any $o, x \in \Omega$,

$$b_p(go, gx) = \lim_{c\to p} d_\Omega(go,c) - d_\Omega(gx,c) = \lim_{c\to p} d_\Omega(go,gc) - d_\Omega(gx,gc) = \lim_{c\to p} d_\Omega(o,c) - d_\Omega(x,c) = b_p(o,x),$$

since, if c tends to p, gc also. Hence, for any $x \in \Omega$,

$$\mathcal{H}_p(gx) = \{y \in \Omega,\ b_p(gx,y) = 0\} = \{y \in \Omega,\ b_p(x, g^{-1}y) = 0\} = gH_p(x),$$

that is, g preserves the set of horospheres based at p. Now, for any $x, y \in \Omega$, we have

$$b_p(x,gx) = b_p(x,y) + b_p(y,gy) + b_p(gy,gx) = b_p(y,gy) := a \in \mathbb{R}.$$

Since $|b_p(x,gx)| \leqslant d_\Omega(x,gx)$, this implies that, for any $x \in \Omega$, $d_\Omega(x,gx) \geqslant |a|$. Since $\tau(g) = 0$, we get $a = 0$, that is, $gx \in \mathcal{H}_p(x)$. $\square$

By a **cusp**, defined by a discrete parabolic subgroup $\mathcal{P}$ fixing p, we will mean the quotient of some horoball based at p by $\mathcal{P}$.

Proposition 1.3.3 ([26])**.** *Let $\Omega \subset \mathbb{RP}^n$ be a strictly convex proper open set with C^1 boundary, and $\mathcal{P}$ a parabolic subgroup of $Isom(\Omega, d_\Omega)$ fixing the point $p \in \partial\Omega$. Then, for any horosphere $\mathcal{H}$ based at p, $\mathcal{H} \setminus \{p\}$, as well as $\partial\Omega \setminus \{p\}$, carries an affine structure preserved by $\mathcal{P}$.*

Proof. The set of lines passing through p is the projective space $P_p = \mathbb{P}(\mathbb{R}^{n+1}/\langle p \rangle)$, of dimension $n-1$. $\mathcal{P}$ acts projectively on this space, and preserves the projective hyperspace T_p consisting of lines tangent to $\partial\Omega$ at p. Thus, $\mathcal{P}$ acts affinely on the affine space $P_p \setminus T_p$, that one can identify with $\partial\Omega \setminus \{p\}$ or $\mathcal{H} \setminus \{p\}$ for any horosphere $\mathcal{H}$ based at p. In this way, we see that each $\mathcal{H} \setminus \{p\}$, as well as $\partial\Omega \setminus \{p\}$, carries an affine structure preserved by $\mathcal{P}$. ☐

1.3.4 Isometries of plane Hilbert geometries

We describe here what occurs in the easy case of plane Hilbert geometries. In dimension 2, isometries of a general Hilbert geometry (Ω, d_Ω) are well classified, see for example [19] or [57]. In particular, if Ω is strictly convex,

- any hyperbolic isometry γ can be represented as a matrix

$$\begin{pmatrix} \lambda_0 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$$

  with $\lambda_0 > \lambda_1 > \lambda_2 > 0$.

- any parabolic isometry γ can be represented by the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

  This implies in particular that the orbit of any point $x \in \mathbb{RP}^2$ under γ lies on a conic which contains the fixed point p of γ. Indeed, if the basis is chosen so that γ has the preceding matrix form, then it preserves the family of conics

$$E_{\lambda,\mu} = \{\lambda z^2 + \mu(y^2 - z(y+2x))\},\ \lambda, \mu \in \mathbb{R}.$$

  The degenerated case $\mu = 0$ is the line $z = 0$, which corresponds to the tangent line to $\partial\Omega$ at p. If $x = [a : b : 1] \in \mathbb{RP}^2 \setminus \{z = 0\}$, we denote by $E_x = E_{b+2a-b^2,1}$ the conic preserved by p and containing x (and its orbit).

Let $p \in \partial\Omega$. $\partial\Omega \setminus \{p\}$ carries an affine structure preserved by any parabolic isometry fixing p. Such an isometry has no fixed point on $\partial\Omega \setminus \{p\}$, hence it acts as a translation on $\partial\Omega \setminus \{p\}$. In particular, the group of parabolic isometries fixing a common point p is isomorphic to a subgroup of $\mathbb{R}$, and any discrete parabolic group is thus isomorphic to $\mathbb{Z}$, generated by some parabolic isometry γ.

So, any discrete parabolic group fixing p acts cocompactly on $\partial\Omega \setminus \{p\}$. Thus, we can find points $x \in \Omega$ and $y \notin \Omega$ such that the ellipses E_x and E_y define two convex sets $\mathcal{E}_x$ and $\mathcal{E}_y$ such that $\mathcal{E}_x \subset \Omega \subset \mathcal{E}_y$ and $E_x \cap E_y = E_x \cap \partial\Omega = E_y \cap \partial\Omega = \{p\}$. Hence the following

Lemma 1.3.4. *Let $\Omega \subset \mathbb{RP}^2$ be a strictly convex proper open set with C^1 boundary. Let $\mathcal{P}$ be a discrete parabolic group of $Isom(\Omega, d_\Omega)$ fixing p. Then $\mathcal{P}$ is isomorphic to $\mathbb{Z}$ and preserves two ellipses E and E' such that $E \cap E' = E \cap \partial\Omega = E' \cap \partial\Omega = \{p\}$ and $\mathcal{E} \subset \Omega \subset \mathcal{E}'$, where $\mathcal{E}$ and $\mathcal{E}'$ are the convex hulls of E and E'.*

An important consequence of this is that, given a sufficiently small horoball based at p, $\mathcal{E}$ and $\mathcal{E}'$ define two hyperbolic metrics $\mathtt{h}$ and $\mathtt{h}'$ on the cusp $H/\mathcal{P}$, such that $\mathtt{h}' \leqslant F \leqslant \mathtt{h}$.

Proposition 1.3.5. *Let $\Omega \subset \mathbb{RP}^2$ be a strictly convex proper open set with C^1 boundary. Let $\mathcal{P}$ be a discrete parabolic subgroup of $Isom(\Omega, d_\Omega)$ fixing $p \in \partial\Omega$. Choose any $C > 1$. Then any sufficiently small horoball H based at p carries two $\mathcal{P}$-invariant hyperbolic metrics $\mathtt{h}$ and $\mathtt{h}'$ such that, on H,*

- *F, $\mathtt{h}$ and $\mathtt{h}'$ have the same geodesics, up to parametrization;*
- $\dfrac{1}{C}\mathtt{h} \leqslant \mathtt{h}' \leqslant F \leqslant \mathtt{h} \leqslant C\mathtt{h}'$

Proof. Choose $\mathcal{E}$ and $\mathcal{E}'$ as in the last lemma. Any sufficiently small horoball H_0 based at p is inside $\mathcal{E}$. $\mathcal{E}$ and $\mathcal{E}'$ then define two $\mathcal{P}$-invariant hyperbolic metrics on H_0, such that $\mathtt{h}' \leqslant F \leqslant \mathtt{h}$. Furthermore, $\mathcal{E}$ is a horoball based at p of the hyperbolic space $\mathcal{E}'$. Now this is obvious in the upper half-space model of $\mathbb{H}^2$ that, for any $C > 1$, we can choose a sufficiently small horoball H' of $\mathcal{E}'$ on which $\mathtt{h}/\mathtt{h}' \leqslant C$. For any horoball H (of Ω) inside this H', we will still have $\mathtt{h}/\mathtt{h}' \leqslant C$ hence the result. □

1.4 Manifolds modeled on Hilbert geometries

We want to consider manifolds modeled on Hilbert geometries (Ω, d_Ω) defined by a strictly convex proper open set Ω with C^1 boundary. Such a manifold M appears as a quotient $M = \Omega/\Gamma$ of Ω by a discrete subgroup Γ of isometries without torsion, that is, Γ does not contain elliptic elements. Since Ω is strictly convex, Γ is a discrete subgroup of the projective group.
Those manifolds are called (strictly) convex projective manifolds. On an abstract smooth manifold of dimension n, a projective structure is an atlas (U_i, φ_i) with coordinate charts with values in the projective space $\mathbb{RP}^n$, such that changes of coordinates are projective maps. Associated to a projective structure are a developing map from the universal cover $\tilde{M}$ to $\mathbb{RP}^n$ and a representation $\Gamma = \rho(\pi_1(M)) < PGL(n+1, \mathbb{R})$ of the fundamental group of M. We say that the projective structure is convex if the developing map is a diffeomorphism onto an convex proper open subset Ω of $\mathbb{RP}^n$, in which case $M = \Omega/\Gamma$.

For a discrete group $\Gamma < PGL(n+1, \mathbb{R})$ acting on Ω, we can always find a locally finite convex fundamental domain, as claimed by the following theorem, due to Lee [50]. A simple proof can be found in [57].
By a fundamental domain for Γ, we mean a subset K of Ω such that $\Gamma.K = \Omega$ and for any two distinct $\gamma, \gamma' \in \Gamma$, $\gamma.K \cap \gamma'.K = \emptyset$. Locally finite means that for any compact subset C of Ω, the number of translates $\gamma.K$ of K that intersect C is finite.

Theorem 1.4.1 (Lee, [50])**.** *Let $\Gamma < PGL(n+1, \mathbb{R})$ be a discrete group acting on a convex proper open set $\Omega \subset \mathbb{RP}^n$. There exists a locally finite convex fundamental domain for the action of Γ on Ω.*

1.4.1 The limit set

If Γ is a discrete group of isometries of (Ω, d_Ω), its **limit set** Λ_Γ is the set of accumulation points of an orbit $\Gamma.o$ in $\partial\Omega$, defined by

$$\Lambda_\Gamma = \overline{\Gamma.o} \smallsetminus \Gamma.o.$$

This definition does not depend on the point o that we consider, thanks to our assumptions on Ω: strict convexity and C^1 boundary.
Λ_Γ is the minimal invariant closed subset of $\partial\Omega$ which is invariant under Γ: any other nonempty Γ-invariant closed subset contains Λ_Γ. In particular, the action of Γ on Λ_Γ is minimal, that is, every orbit is dense.
Obviously, Λ_Γ contains the set of all the fixed points F of the elements of Γ in $\partial\Omega$. The closure $\overline{F}$ of F being Γ-invariant, we conclude that $\overline{F} = \Lambda_\Gamma$.
We say that a discrete group Γ of isometries is **elementary** if its limit set is finite. It can then consist of 0, 1 or 2 points, which correspond respectively to the following cases: Γ is elliptic[3], Γ is parabolic, $\Gamma = \langle h \rangle$ is the cyclic group generated by a hyperbolic isometry.
When Γ is neither an elliptic or a parabolic elementary group, Λ_Γ is indeed the closure of the set of fixed points of hyperbolic isometries. This is just the fact that a nonelementary group contains necessarily a hyperbolic isometry.

The same group $\Gamma < PGL(n+1,\mathbb{R})$ can act on various convex sets Ω. For example, it acts on the convex hull $C(\Lambda_\Gamma)$ of its limit set. In fact, $C(\Lambda_\Gamma)$ is the smallest convex set on which Γ can act. Remark that $C(\Lambda_\Gamma)$ is not necessarily open in $\mathbb{RP}^n$: for instance, the limit set of a parabolic subgroup is reduced to one point, hence $C(\Lambda_\Gamma)$ also. These remarks, though naive, are crucial when we consider the problem of understanding the properties of an eventual quotient Ω/Γ, when the discrete subgroup Γ of $PGL(n+1,\mathbb{R})$ is given, and not the convex set Ω.

1.4.2 Compact quotients

We say that a convex proper open set Ω is **divisible** if it admits a compact quotient by some discrete projective group. The ellipsoid is the only divisible strictly convex set which is homogeneous, that is, with a transitive group of isometries. The existence of other divisible strictly convex sets is nontrivial. The first example was given by Kac and Vinberg [43] in 1967, using Coxeter groups.
In 1984, Johnson and Millson [42] constructed examples of hyperbolic manifolds in all dimensions, whose fundamental group $\Gamma_0 \subset Isom(\mathbb{H}^n)$ could be deformed continuously into Zariski-dense subgroups Γ_t of $SL(n+1,\mathbb{R})$. The work of Koszul [48] implies that such little deformations still divide some convex sets Ω_t; since (Ω_t, d_{Ω_t}) is quasi-isometric to the Gromov-hyperbolic group Γ_t, (Ω_t, d_{Ω_t}) is itself Gromov-hyperbolic, so in particular strictly convex (see Benoist's theorem below).
In dimension 2, a very precise description has been given by Goldman [35]. He proved in particular that the deformation space of convex projective structures on the surface Σ_g of genus $g \geqslant 2$ is a manifold diffeomorphic to $\mathbb{R}^{16g-16}$; it contains the Teichmüller space of Σ_g as a submanifold of dimension $6g-6$. Such a description is not available in higher dimensions, except for Marquis' work [54].

The main general result about the geometry of divisible convex sets is the following theorem of Benoist. It divides the set of divisible convex sets into two families and only one of them, which

[3] Γ is said to be elliptic if all its elements but the identity are elliptic isometries fixing the same point.

includes ellipsoids, is studied in this thesis,. The last property, which is an intrinsic property of the abstract fundamental group of the quotient manifold, implies that a manifold M cannot carry a strictly convex projective structure and a nonstrictly convex one.

Theorem 1.4.2 ([7]). *Let $\Omega \subset \mathbb{RP}^n$ be a convex proper open set, which can be divided by some discrete group $\Gamma < PGL(n+1,\mathbb{R})$. The following statements are equivalent:*

- *Ω is strictly convex;*
- *the boundary $\partial\Omega$ of Ω is C^1;*
- *the space (Ω, d_Ω) is Gromov-hyperbolic;*
- *the group Γ is Gromov-hyperbolic.*

If all these results show that strictly convex projective structures are far more general than hyperbolic ones, the examples that were given above are all deformations of hyperbolic structures. That is always the case in dimensions 2 and 3, but quite surprisingly, in dimension higher than 4, there exist compact manifolds which admit strictly convex projective structures but no hyperbolic one. Such examples were first constructed by Benoist [8] in dimension 4, 5 and 6, using Coxeter groups; Kapovich [46] then proved that some of the manifolds constructed by Gromov and Thurston in [38] were providing other examples, in all dimensions.

1.4.3 Geometrically finite quotients

We extend here the notion of geometrical finiteness to our context, as well as some results which are essential for studying the dynamics.

Definitions 1.4.3. *Let Γ be a discrete group of isometries of (Ω, d_Ω). A point $p \in \Lambda_\Gamma$ is said to be*

- **radial** *or* **conical** *if there exist a point $o \in \Omega$ and a sequence of isometries (g_n) in Γ, such that the sequence $(g_n o)$ converges to p in $\overline{\Omega}$ and*
$$\sup_n d_\Omega(g_n o, [op]) < +\infty;$$
- **a bounded parabolic** *point if p is the fixed point of a parabolic subgroup $\mathcal{P}$ of Γ which acts cocompactly on $\Lambda_\Gamma \setminus \{p\}$.*

The following geometrical characterization of conical points will be often used:

Remark 1.4.4. *A point $p \in \Lambda_\Gamma$ is conical if and only if, for any point $x \in \Omega$, the projection on the quotient $M = \Omega/\Gamma$ of the ray $[xp)$ ending at p stays in a compact part K of M an infinite period of time.*

Definition 1.4.5. *Let $M = \Omega/\Gamma$ be the quotient manifold of a strictly convex set Ω with C^1 boundary. M is said to be* **geometrically finite** *if Λ_Γ consists only of radial and bounded parabolic points.*

48 (This definition is the one used in hyperbolic geometry. As we pointed out in [26], it is not sufficient in Hilbert geometry if we want the quotient manifold to satisfy a reasonable "geometrical finiteness". But there is no difference in dimension 2.)
The goal of what follows is to prove theorem 1.4.8, which describes the convex core of a geometrically finite manifold. The **convex core** $C(M)$ of M is defined as the closure (in M) of the quotient $C(\Lambda_\Gamma)/\Gamma \subset M = \Omega/\Gamma$, where $C(\Lambda_\Gamma) \subset \mathbb{RP}^n$ denotes the (open) convex hull of the limit set Λ_Γ. The description provided by theorem 1.4.8 is fundamental because the recurrent part of the dynamics occurs in the convex core (see section 2.5).

Lemma 1.4.6. *Let Γ be a discrete group of isometries of (Ω, d_Ω). A parabolic point of Λ_Γ is not conical.*

Proof. Let p be a parabolic point fixed by the parabolic isometry γ. If p is conical, from remark 1.4.4, we can find some $x \in \Omega$ such that the ray $[xp)$ gives on the quotient a geodesic ray that stays in a compact part K of M an infinite period of time. Consider the function $t \longmapsto d_\Omega([xp)(t), [\gamma(x)p)(t))$ that represents in Ω the distance between the ray $t \mapsto [xp)$ and its image by γ. Since $\partial\Omega$ is C^1, this function decreases to 0. But the injectivity radius of the compact part K is > 0, hence a contradiction. □

Lemma 1.4.7. *Let $M = \Omega/\Gamma$ be a geometrically finite manifold and D a closed convex fundamental domain for Γ on $\overline{C(\Lambda_\Gamma)}$. Then any connected component of $D \cap \Lambda_\Gamma$ consists of a single parabolic point.*

Proof. Let p be a point in $D \cap \Lambda_\Gamma$. If x is any point in D, the projection on M of the ray $[xp)$ leaves any compact set, hence p is not conical, by remark 1.4.4. Since Γ is geometrically finite, p is necessarily parabolic. Let γ be a parabolic element fixing p. γ acts bijectively on the connected component C of p in Λ_Γ. Now, we know from theorem 1.3.1 that, for any point $q \in \overline{\Omega}$, the sequence $(\gamma^n q)$ tends to p, which implies that $C = \{p\}$. □

We can now prove the main

Theorem 1.4.8 ([26]). *Let $M = \Omega/\Gamma$ be a geometrically finite manifold. Then the number of conjugacy classes of maximal parabolic subgroups of Γ is finite and the convex core of M can be decomposed as the disjoint union*

$$K \bigsqcup \sqcup_{k=1}^{l} C_k$$

of a compact part K and a finite number of cusps C_k, each cusp corresponding to a conjugacy class of maximal parabolic subgroups.

(This result is true under the stronger notion of geometrical finiteness introduced in [26].)

Proof. For any parabolic point $p \in \Lambda_\Gamma$, let $\mathcal{P}_p = \mathrm{Stab}_\Gamma(p)$ be the maximal parabolic subgroup fixing it. Let D be a locally finite convex closed fundamental domain for Γ on $\overline{C(\Lambda_\Gamma)}$ and pick a parabolic point $p \in D \cap \Lambda_\Gamma$. We can find a closed fundamental domain C for $\mathcal{P}_p$ on $\overline{C(\Lambda_\Gamma)}$ that contains D; since p is bounded, $C \cap \Lambda_\Gamma \setminus \{p\}$ is compact in $\Lambda_\Gamma \setminus \{p\}$. The set $D \cap \Lambda_\Gamma \setminus \{p\}$ consisting of parabolic points is contained in the compact $C \cap \Lambda_\Gamma \setminus \{p\}$, so $D \cap \Lambda_\Gamma$ is discrete in Λ_Γ, hence finite. Choose parabolic points $p_1, \cdots, p_l$ in D, such that any two $\mathcal{P}_{p_i},\ i = 1, \cdots, l$, are not conjugated. We can then put disjoint horoballs $H_{p_1}, \cdots, H_{p_l}$ based at these points, and the set $\Gamma\{H_{p_i},\ 1 \leqslant i \leqslant l\}$ consists of disjoint horoballs based at parabolic points. Let

$$C_i = H_{p_i}/\mathcal{P}_{p_i} = \Gamma.H_{p_i}/\Gamma,\ \overline{C_i} = H_{p_i} \cup \{p_i\}/\mathcal{P}_{p_i} = \Gamma.(H_{p_i} \cup \{p_i\})/\Gamma,$$

for $1 \leqslant i \leqslant l$, and

$$K = \overline{C(\Lambda_\Gamma)}/\Gamma \smallsetminus \cup_{i=1}^{l} \overline{C_i}.$$

Each $\overline{C_i}$ is open in the compact $\overline{C(\Lambda_\Gamma)}/\Gamma$, so K is closed in $\overline{C(\Lambda_\Gamma)}/\Gamma$, hence compact. (I guess the mistake is here !) This yields the decomposition.
Now, let p be any parabolic point in Λ_Γ and pick a geodesic ray (xp) inside $C(\Lambda_\Gamma)$, that is, such that $x \in C(\Lambda_\Gamma)$. Since p is not conical, the corresponding geodesic ray on the quotient $M = \Omega/\Gamma$ leaves any compact subset, hence is ultimately contained in a cusp C_i. Thus there are some $i \in \{1, \cdots, l\}$ and $\gamma \in \Gamma$ such that $\gamma.p = p_i$, that is $\mathcal{P}_p$ is conjugated to $\mathcal{P}_{p_i}$. The number of conjugacy classes of maximal parabolic subgroups is thus finite, equal to the number of cusps of M. □

1.4.4 The case of surfaces

Geometrically finite surfaces

For surfaces, we can easily go further because we are able to describe parabolic subgroups, hence the Hilbert geometry of a cusp. We set the results in the following corollary, which is a direct consequence of proposition 1.3.5 and theorem 1.4.8.

Corollary 1.4.9. *Let $M = \Omega/\Gamma$ be a geometrically finite surface. Then, for any $C > 1$, there exists a decomposition of $C(M)$ into*

$$M = K \bigsqcup \sqcup_{i=1}^{l} C_i$$

consisting of a compact part K and a finite number of cusps C_i, on which there exist hyperbolic metrics $\mathtt{h}_i$ and $\mathtt{h}'_i$ such that

- *F, $\mathtt{h}_i$ and $\mathtt{h}'_i$ have the same geodesics on C_i, up to parametrization;*
- $\dfrac{1}{C}\mathtt{h}_i \leqslant \mathtt{h}'_i \leqslant F \leqslant \mathtt{h}_i \leqslant C\mathtt{h}'_i$.

Finite volume surfaces

Marquis' description of finite volume surfaces can go as follow.

Theorem 1.4.10 (Marquis, [57])**.** *Let $\Omega \subset \mathbb{RP}^2$ be a convex proper open set. A surface $M = \Omega/\Gamma$ has finite volume if and only if M is geometrically finite and $\Lambda_\Gamma = \partial\Omega$.*

To understand this statement, we have to explain what we mean by volume. Indeed, a Finsler geometry has no canonical volume as a Riemannian one. The volume that we use here is the so called **Busemann volume**, which corresponds to the Hausdorff measure of the metric d_Ω (see [18]). This volume *vol* is defined by renormalizing any volume on Ω in such a way that each tangent unit ball has volume one. More precisely, let λ be any Lebesgue measure on Ω. We define

$$dvol(x) = \frac{d\lambda(x)}{\lambda(B_x(1))},$$

where $B_x(1) = \{v \in T_x\Omega,\ F(x,v) = 1\}$ is the tangent unit ball for $F(x,.)$. This construction provides a volume on any quotient manifold of Ω. Finite volume manifolds are considered with respect to this volume.

The construction of the Busemann volume can be made for any Finsler manifold. In particular, we can define a volume on any C^1 submanifold of Ω. This remark will be used in the proof of the Ruelle inequality, in section 5.2.1.

Remark that, if $\Omega \subset \Omega'$ are two convex proper open subsets of $\mathbb{RP}^n$, then the Busemann volumes vol and vol' on Ω and Ω' satisfy $vol \geqslant vol'$ on Ω. That yields the following

Lemma 1.4.11. *Let $\Omega \subset \mathbb{RP}^2$ be a strictly convex proper open set with C^1 boundary. Let $\mathcal{P}$ be a discrete parabolic subgroup fixing $p \in \partial\Omega$ and H be any horoball based at p. Then $H/\mathcal{P}$ has finite volume.*

Proof. Consider two $\mathcal{P}$-invariant convex sets $\mathcal{E}$ and $\mathcal{E}'$ as in lemma 1.3.4. Let H be any horoball based at p. Since $\mathcal{P}$ acts cocompactly on $\partial H \smallsetminus \{p\}$, we can assume that $H \subset \mathcal{E}$. From hyperbolic geometry, we know that $vol^{\mathcal{E}}(H/\mathcal{P})$ is finite, where $vol^{\mathcal{E}}$ denotes the hyperbolic volume defined by the hyperbolic space $\mathcal{E}$. Since $vol \leqslant vol^{\mathcal{E}}$, we get the result. ☐

As a corollary, we get that the convex core of a geometrically finite surface has finite volume. Hence we get the if part of theorem 1.4.10. For more details, we refer to [57].

1.5 Volume entropy

The volume entropy of a Riemannian metric g on a manifold M measures the asymptotic exponential growth of the volume of balls in the universal cover $\tilde{M}$; it is defined by

$$h_{vol}(g) = \limsup_{R\to+\infty} \frac{1}{r} \log vol_g(B(x,R)), \tag{1.3}$$

where vol_g denotes the Riemannian volume corresponding to g. We define the volume entropy of a Hilbert geometry (Ω, d_Ω) by the same formula, with respect the Busemann volume.

It is not clear when the limit in (1.3) exists, but some results are already known: as a consequence of theorem 1.1.2, if Ω is a polytope then $h_{vol}(\Omega, d_\Omega) = 0$; at the opposite, we have the

Theorem 1.5.1 ([10])**.** *Let $\Omega \subset \mathbb{RP}^n$ be a convex proper open set. If the boundary $\partial\Omega$ of Ω is $C^{1,1}$, that is, has Lipschitz derivative, then $h_{vol}(\Omega, d_\Omega) = n - 1$.*

The global feeling is that any Hilbert geometry is in between the two extremal cases of the ellipsoid and the simplex. In particular, the following conjecture is still open.

Conjecture 1.5.2. *For any $\Omega \subset \mathbb{RP}^n$,*

$$h_{vol}(\Omega, d_\Omega) \leqslant n - 1.$$

In [10] the conjecture is proved in dimension $n = 2$ and an example is explicitly constructed where $0 < h_{vol} < 1$.

Remark that in the case of a convex set Ω divided by Γ, we can choose any volume on the quotient manifold Ω/Γ, or even any probability measure and lift it to Ω. The volume entropy does not depend on the choice of such a measure. In particular, by choosing a Dirac measure, it is the same as looking at the exponential growth rate of the orbit of a point $o \in \Omega$ under Γ. This number is the critical exponent of Γ:

$$\delta_\Gamma = \limsup_{R\to+\infty} \frac{1}{R} \log N_\Gamma(o, R),$$

where

$$N_\Gamma(o, R) = \sharp\{\gamma,\ d_\Omega(o, \gamma o) < R\}$$

denotes the number of points of the orbit $\Gamma.o$ in the ball of radius R about o. This quantity is the main character of the last two chapters.

For a nonnecessarily cocompact group, the volume entropy is in general bigger than the critical exponent:

$$h_{vol} \geqslant \delta_\Gamma.$$

For example, in the hyperbolic space, $h_{vol} = n - 1$, but δ_Γ clearly depends on the group. Take for example a punctured torus. The loop γ around the puncture can be represented by a parabolic or a hyperbolic element of $Isom(\mathbb{H}^2)$. In the first case, the surface has finite volume and $h_{vol} = \delta_\Gamma = 1$; in the second one, it is convex cocompact, and we can change the length of the geodesic loop representing γ so that δ_Γ can take any value in $(0, 1)$.

1.6 Topological entropy

1.6.1 The compact case

A major invariant in the theory of dynamical systems is the topological entropy, which roughly speaking measures how the system separates the points, how much it is chaotic.

Given a flow $\varphi^t : X \longrightarrow X$ on a compact metric space (X, d), we define the distances $d_t,\ t \geqslant 0$, on X by $d_t(x, y) = \max_{0\leqslant s\leqslant t}\ d(\varphi^s(x), \varphi^s(y)),\ x, y \in X$. The topological entropy of φ is then the well defined quantity

$$h_{top}(\varphi) = \lim_{\epsilon\to 0} \left[\limsup_{t\to+\infty} \frac{1}{t} \log N(\varphi, t, \epsilon) \right] \in [0, +\infty],$$

where $N(\varphi, t, \epsilon)$ denotes the minimal number of balls of radius ϵ for d_t needed to cover X.

In [51], Anthony Manning proved the following result:

Theorem 1.6.1. *Let (M, g) be a compact Riemannian manifold of volume entropy h_{vol}. Let h_{top} be the topological entropy of the geodesic flow of g on HM. We always have*

$$h_{top} \geqslant h_{vol}.$$

Furthermore, if the sectional curvature of M is negative then

$$h_{top} = h_{vol}.$$

In his PhD thesis, Daniel Egloff [32] extended this result for some regular Finsler manifolds. In fact, Manning's proof still works in the special case we are dealing with here to get the

Proposition 1.6.2. *Let $\Omega \subset \mathbb{RP}^n$ be a strictly convex proper open set, and $M = \Omega/\Gamma$ a compact manifold modeled on Ω. Then*

$$h_{top} = h_{vol} = \delta_\Gamma.$$

We do not reproduce the proof here, see [51]. The only point we have to check is the following technical lemma that Manning proved using negative curvature. Here we can compute it directly.

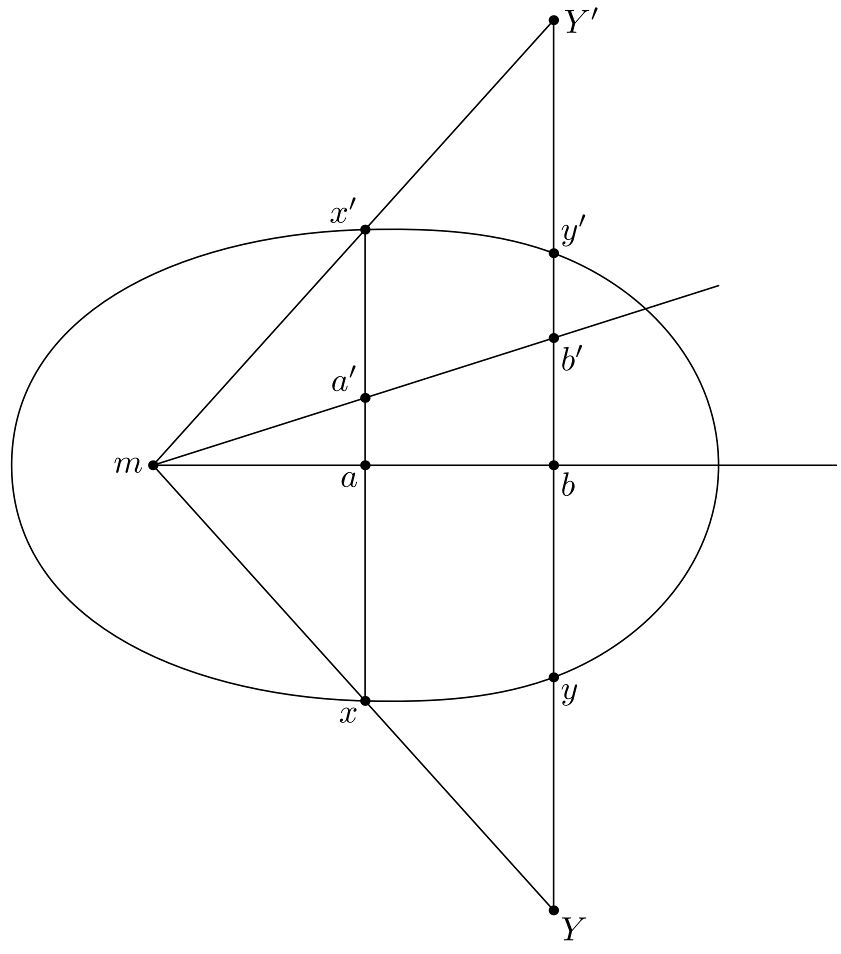

Figure 1.7: To follow the proof of lemma 1.6.3

Lemma 1.6.3. *The distance between corresponding points of two geodesics $\sigma, \tau : [0, r] \to \Omega$ is at most $d_\Omega(\sigma(0), \tau(0)) + d_\Omega(\sigma(r), \tau(r))$.*

Proof. There are two cases: either σ and τ meet each other or not. Anyway, by joining the point $\sigma(0)$ and $\tau(r)$ with a third geodesic, we see we only have to prove that the distance between two different lines going away from the same point (but not necessary with the same speed) increases. So suppose $c, c' : \mathbb{R} \to \Omega$ are two lines beginning at the same point $m = c(0) = c'(0)$. Take two pairs of corresponding points $(a, a') = (c(t_1), c'(t_1)), (b, b') = (c(t_2), c'(t_2))$ with $t_2 > t_1 \geqslant 0$. We want to prove that $d_\Omega(a, a') < d_\Omega(b, b')$. As it is obvious if $t_1 = 0$, assume $t_1 > 0$ and note x, x' and y, y' the points on the boundary $\partial\Omega$ of Ω such that x, a, a', x' and y, b, b', y' are on the same line, in this order. Note also $Y = (mx) \cap (bb')$ and $Y = (mx') \cap (bb')$, so that by convexity of Ω, the six points Y, y, b, b', y', Y' are different and on the same line, in this order. The two lines (aa') and (bb') meet at a certain point that we can send at infinity by an homography. So we can assume the two lines are parallel (c.f. figure 1.7).
Then it follows from Thales' theorem that

$$1 > [x, a, a', x] = [Y, b, b', Y'] > [y, b, b', y'],$$

so that

$$d_\Omega(a, a') = |\log([x, a, a', x])| < |\log([y, b, b', y'])| = d_\Omega(b, b').$$

□

1.6.2 The noncompact case

Consider the system defined by a flow $\varphi^t : X \longrightarrow X$ of a nonnecessarily compact metric space (X, d). Bowen [17] extended the definition of topological entropy to this setting. It consists in exhausting the space by compact subsets, compute the entropy of each such set and take the supremum.

More precisely, if K is any compact subset of X, we can look at the spaces (K, d_t) for $t \geqslant 0$, where the distances d_t are defined as in the last section. The topological entropy of φ on (K, d) is defined by

$$h_{top}(\varphi, d) = \lim_{\epsilon \to 0} \left[\limsup_{t \to +\infty} \frac{1}{t} \log N_{(K,d)}(\varphi, t, \epsilon) \right] \in [0, +\infty],$$

where $N_{(K,d)}(\varphi, t, \epsilon)$ denotes the minimal number of balls of radius ϵ for d_t needed to cover K. The topological entropy of φ on (X, d) is then

$$h_{top}(\varphi, d) = \sup_K h_{top}(K, d),$$

where the supremum is taken over all compact subsets of X.

In the case of a noncompact space, this quantity may depend on the distance d, since all the distances defining the same topology on X are not equivalent. To make it independant on the distance, we then take the infimum on all the distances which define the same topology. In formula, the topological entropy of φ on X is defined as

$$h_{top}(\varphi) = \inf_d h_{top}(\varphi, d).$$

As shown by Handel and Kitchens in [39], this generalization seems to be the good one.

In the context of this thesis, we will see that the topological entropy of the geodesic flow on a noncompact quotient Ω/Γ is actually equal to the critical exponent δ_Γ of the group Γ. This is the goal of section 5.1.

Chapter 2

Dynamics of the geodesic flow

We describe here the main tool that we will use to study the geodesic flow of Hilbert metrics. The last section proves the uniform hyperbolicity of the geodesic flow on a compact quotient and a geometrically finite surface.

2.1 Foulon's dynamical formalism

Here we explain how to extend to the context of this work the dynamical objects introduced by Patrick Foulon in [33] to study second-order differential equations: they provide analogues of Riemannian objects such as covariant differentiation, parallel transport and curvature for any such equation which is regular enough.
We want to apply that formalism to our Hilbert geometries, which are more irregular. Here we carefully check that these objects are still well defined, and even smooth in some sense, under some specific assumptions. For more details about this, we refer the reader to [33] and to the appendix of [34] for an English version.

This part is also introducing some notations that will be used all along the text.

2.1.1 Directional smoothness

Assume a smooth vector field X^0 is given on a smooth manifold W. We denote by

- $C_{X^0}(W)$ (or simply C_{X^0}) the set of functions f on W such that, for any $n \geqslant 0$, $L^n_{X^0} f$ exists;
- $C^p_{X^0}(W)$ (or simply $C^p_{X^0}$) the set of functions $f \in C_{X^0}$ such that, for any $n \geqslant 0$, $L^n_{X^0} f \in C^p(W)$.

A C_{X^0} (respectively $C^p_{X^0}$) vector field Z will be a section of $W \longrightarrow TW$ which is smooth in the direction X^0, that is, the Lie derivative $L^n_{X^0} Z$ exists (respectively exists and is C^p) for any $n \geqslant 0$.
Equivalently, Z can be locally written as $Z = \sum f_i Z_i$ where the Z_i are smooth vector fields on W, and $f_i \in C_{X^0}$ (respectively $f_i \in C^p_{X^0}$).
When X^0 is a complete vector field, f being in C_{X^0} means that f is smooth all along the orbits of the flow generated by X^0.

Lemma 2.1.1. *Let $m \in C^1_{X^0}$ and $X = mX^0$. For any C_{X^0} vector field Z,*

(i) $L_Z m \in C_{X^0}$;

(ii) for any $n \geqslant 0$, the Lie derivative $L^n_X Z = [X[\cdots[X,Z]\cdots]$ is a C_{X^0} vector field.

In some sense, if $X = mX^0$ with $m \in C^1_{X^0}$,this lemma means that to be smooth with respect to X is equivalent to being smooth with respect to X^0. The proof will make use of the following improved version of Schwartz' theorem.

Lemma 2.1.2. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a C^1 map. If $\frac{\partial^2 f}{\partial x_i \partial x_j}$ exists and is continuous then so is $\frac{\partial^2 f}{\partial x_j \partial x_i}$ and we have $\frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j}$.*

Proof of lemma 2.1.1. (i) Let $w \in W$. Since X^0 is smooth, we can find smooth coordinates $(x_0, x_1, \cdots, x_n)$ on a neighbourhood V_w of w such that $X^0 = \frac{\partial}{\partial x_0}$ and $Z = \sum z_i X^i$, where $z_i \in C_X(V_w)$ and $X^i = \frac{\partial}{\partial x_i}$.
Let $f \in C^1_{X^0}$. Then on V_w, we formally have

$$L_{X^0} L_Z f = \sum L_{X^0}(z_i L_{X^i} f) = \sum L_{X^0} z_i L_{X^i} f + \sum z_i L_{X^0}(L_{X^i} f).$$

In fact, this expression makes sense. The first term is well defined and in C_{X^0}. The second one exists from lemma 2.1.2; we even have $L_{X^0} L_{X^i} f = L_{X^i} L_{X^0} f$, so that

$$L_{X^0} L_Z f = L_Z L_{X^0} f + \sum L_{X^0} z_i L_{X^i} f. \tag{2.1}$$

We now prove that $L^n_{X^0} L_Z m$ exists by induction on n. Assume that for some $n \geqslant 0$, we know that

$$L^n_{X^0} L_Z m = m_n + L_Z L^n_{X^0} m$$

for some function $m_n \in C_{X^0}$. Then

$$L^{n+1}_{X^0} L_Z m = L_{X^0} m_n + L_{X^0} L_Z (L^n_{X^0} m).$$

But $L^n_{X^0} m \in C^1_{X^0}$, so that we can apply the preceding result (equation (2.1)) with $f = L^n_{X^0} m$ to get

$$L_{X^0} L_Z (L^n_{X^0} m) = L_Z L^{n+1}_{X^0} m + g$$

for some function $g \in C_{X^0}$. We thus have

$$L^{n+1}_{X^0} L_Z m = m_{n+1} + L_Z L^{n+1}_{X^0} m$$

with $m_{n+1} = L_{X^0} m_n + g \in C_{X^0}$. That proves the first point.

(ii) The Lie derivative $Z^0_n := L_{X^0} Z$ exists for any $n \geqslant 0$. Let $Z_0 := Z$ and (formally) $Z_n := L^n_X Z$ for $n \geqslant 1$. Assume that for some $n \geqslant 0$, Z_n exists and can be written

$$Z_n = m^n Z^0_n + z_n$$

where z_n is some C_{X^0} vector field. Then

$$\begin{aligned} Z_{n+1} = [X, Z_n] &= m[X^0, m^n Z_n^0 + z_n] - L_{Z_n} m\, X^0 \\ &= m[X^0, z_n] + m^{n+1} Z_{n+1}^0 + n m^n L_{X^0} m\, Z_{n+1}^0 - L_{Z_n} m\, X^0, \end{aligned}$$

so that

$$Z_{n+1} = m^{n+1} Z_{n+1}^0 + z_{n+1}$$

with $z_{n+1} \in C_{X^0}$. That proves the second point. $\square$

2.1.2 Second-order differential equations

Let M be a smooth manifold. The homogeneous tangent bundle

$$\pi : HM = TM \setminus \{0\}/\mathbb{R}_+^* \longrightarrow M$$

of M consists of pairs $(x, [\xi])$ with $x \in M$ and $[\xi] = \mathbb{R}_+^*.\xi,\ \xi \in T_x M \setminus \{0\}$. Call

$$\begin{array}{rccc} r: & TM \setminus \{0\} & \longrightarrow & HM \\ & (x, \xi) & \longmapsto & (x, [\xi]) \end{array}$$

the projection from $TM \setminus \{0\}$ to HM.

Definition 2.1.3 (Foulon, [33]). *A second-order differential equation on M is a vector field $X : HM \longrightarrow THM$ on the homogeneous tangent bundle such that*

$$r \circ d\pi \circ X = Id_{HM}.$$

In what follows, M is a smooth manifold and X a complete C^1 second-order differential equation on M. We make the assumption that $X = mX^0$ where

- X^0 is a smooth second-order differential equation on M;
- $m \in C_{X^0}^1(HM)$.

Lemma 2.1.1 claims that to be smooth with respect to either X or X^0 is equivalent, so we will not make any difference between C_X and C_{X^0} functions or vector fields.
We denote by $(\varphi^t)_{t \in \mathbb{R}}$ the flow generated by X. If $w \in HM$, $\varphi.w$ denotes the orbit of w under the flow φ^t, that is, $\varphi.w = \{\varphi^t(w),\ t \in \mathbb{R}\}$. Remark that X and X^0 have the same orbits, up to parametrization. We follow the presentation made in [33].

2.1.3 The vertical distribution and operator

The vertical distribution is the smooth distribution $VHM = \ker d\pi$ where $\pi : HM \longrightarrow M$ is the bundle projection. The letter Y will always denote a C_X vertical vector field, and we write $Y \in VHM$. The following lemma is proved in [33]:

Lemma 2.1.4. *Let $w_0 \in HM$, $Y_1, \cdots, Y_{n-1}$ be vertical vector fields along $\varphi.w_0$ such that, for any $w \in \varphi.w_0$, $Y_1(w), \cdots, Y_{n-1}(w)$ is a basis of $V_w HM$. Then for any $w \in \varphi.w_0$, the family*

$$X(w), Y_1(w), \cdots, Y_{n-1}(w), [X, Y_1](w), \cdots, [X, Y_{n-1}](w)$$

is a basis of $T_w HM$.

This lemma allows us to define the vertical operator as the C_X-linear operator such that

$$v_X(X) = v_X(Y) = 0 \;\; v_X([X, Y]) = -Y.$$

By C_X-linear, we mean that, for any function $f \in C_X$,

$$v_X(fZ) = f v_X(Z).$$

From the very definition, we can check that

$$v_X = m v_{X^0}. \tag{2.2}$$

2.1.4 The horizontal operator and distribution

The horizontal operator $H_X : VHM \longrightarrow THM$ is the C_X-linear operator defined by

$$H_X(Y) = -[X, Y] - \frac{1}{2} v_X([X, [X, Y]]).$$

Lemma 2.1.1 assures us that this definition makes sense. More precisely, we have

$$[X, Y] = m[X^0, Y] - L_Y m X^0$$

and

$$[X, [X, Y]] = m^2 [X^0, [X^0, Y]] + L_X m [X^0, Y] - (L_X L_Y m - m L_{[X,Y]} m) X^0.$$

Since $v_X = m v_{X^0}$, we thus get

$$H_X(Y) = m H_{X^0}(Y) + L_Y m X^0 + \frac{1}{2} L_{X^0} m Y. \tag{2.3}$$

The horizontal distribution $h^X HM$ is defined by

$$h^X HM = H_X(VHM).$$

The verticality lemma 2.1.4 implies that H_X is injective, so that we get the continuous decomposition

$$THM = \mathbb{R}.X \oplus VHM \oplus h^X HM.$$

By a horizontal vector field $h \in h^X HM$, we will mean a C_X section h of $HM \longrightarrow h^X HM$.

The operators v_X and H_X exchange VHM and $h^X HM$ in the following sense: lemma 2.1.1 allows us to consider the compositions $v_X \circ H_X$ and $H_X \circ v_X$, and see that for any $Y \in VHM$, $h \in h^X HM$,

$$v_X \circ H_X(Y) = Y, \;\; H_X \circ v_X(h) = h. \tag{2.4}$$

In particular, remark that any horizontal vector field h can be written $h = H_X(Y)$, for a unique $Y \in VHM$.

Finally, we can define a pseudo-complex structure

$$J^X : h^X HM \oplus VHM \longrightarrow h^X HM \oplus VHM$$

by setting $J^X = v_X$ on $h^X HM$ and $J^X = -H_X$ on VHM. Equation (2.4) gives

$$J^X \circ J^X = -Id|_{VHM \oplus h^X HM}.$$

2.1.5 Projections

We associate to the decomposition

$$THM = \mathbb{R}.X \oplus VHM \oplus h^X HM$$

the corresponding decomposition of the identity:

$$Id = p^X \oplus p_v^X \oplus p_h^X.$$

We immediately have that

$$p_h^X = H_X \circ v_X. \tag{2.5}$$

Moreover,

Lemma 2.1.5. *For any C_X vector field Z, we have*

$$p^X(Z) = p^{X^0}(Z) - (L_{v_{X^0}(Z)} \log m) X^0;$$

$$p_v^X(Z) = p_v^{X^0}(Z) - \frac{1}{2}(L_{X^0} \log m) v_{X^0}(Z);$$

$$p_h^X(Z) = p_h^{X^0}(Z) + (L_{v_{X^0}(Z)} \log m) X^0 + \frac{1}{2}(L_{X^0} \log m) v_{X^0}(Z).$$

In particular, every projection of Z is still a C_X vector field.

Proof. Let $Z = aX + Y + h = a^0 X^0 + Y^0 + h^0$ be the two decompositions of the vector field Z along $\varphi.w$. If we let $y = v_{X^0}(h^0) = v_{X^0}(Z)$, we have, using (2.3),

$$h = H_X(v_X(Z)) = \frac{1}{m} H_X(y) = H_{X^0}(y) + \frac{1}{2m} L_{X^0} m \; y + \frac{1}{m} L_y m \; X^0.$$

Thus

$$h = h^0 + \frac{1}{2} L_{X^0}(\log m) y + L_y(\log m) X^0,$$

and

$$Z = (aX + L_y(\log m) X^0) + (Y + \frac{1}{2} L_{X^0}(\log m) y) + h^0 = a^0 X^0 + Y^0 + h^0.$$

Identifying gives the result. $\square$

2.1.6 Dynamical derivation and parallel transport

We define an analog of the covariant derivation along X that we call the dynamical derivation and denote by D^X. It is the C_X-differential operator of order 1 defined by

$$D^X(X) = 0,\ D^X(Y) = -\frac{1}{2}v_X([X,[X,Y]]),\ [D^X, H_X] = 0.$$

In our context, being a C_X-differential operator of order 1 means that for any $f \in C_X$,

$$D^X(fZ) = fD^X(Z) + (L_X f)Z.$$

Remark that, on VHM, we can write

$$D^X(Y) = H_X(Y) + [X,Y]. \tag{2.6}$$

We can also check that

$$D^X = mD^{X^0} + \frac{1}{2}(L_{X^0}m)Id. \tag{2.7}$$

A vector field Z is said to be parallel along X, or along any orbit, if $D^X(Z) = 0$. This allows us to consider the parallel transport of a C_X vector field along an orbit: given $Z(w) \in T_wHM$, the parallel transport of $Z(w)$ along $\varphi.w$ is the parallel vector field Z along $\varphi.w$ whose value at w is $Z(w)$; the parallel transport of $Z(w)$ at $\varphi^t(w)$ is the vector $Z(\varphi^t(w)) = T^t(Z(w)) \in T_wHM$. Since D^X commutes with J^X, the parallel transport also commutes with J^X. If X is the generator of a Riemannian geodesic flow, the projection on the base of this transport coincides with the usual parallel transport along geodesics.

We can relate the parallel transports with respect to X^0 and X, as stated in the next lemma. This lemma is essential in this work and will be used in many different parts.

Lemma 2.1.6. *Let $w \in HM$ and pick a vertical vector $Y(w) \in V_wHM$. Denote by Y and Y^0 its parallel transports with respect to X and X^0 along $\varphi.w$. Let $h = J^X(Y)$ and $h^0 = J^{X^0}(Y^0)$ be the corresponding parallel transports of $h(w) = J^X(Y(w))$ and $h^0(w) = J^{X^0}(Y^0(w))$ along $\varphi.w$. Then*

$$Y = \left(\frac{m(w)}{m}\right)^{1/2} Y^0$$

and

$$h = -L_Y m\ X^0 + (m(w)m)^{1/2}\ h^0 - \frac{m(w)}{m}\ L_{X^0}m\ Y^0.$$

Proof. We look for the unique vector field Y along $\varphi.w$ such that $D^X(Y) = 0$ and which takes the value $Y(w)$ at the point w. Equation (2.7) gives

$$D^X(Y) = mD^{X^0}(Y) + \frac{1}{2}L_X(\log m)Y.$$

Assume we can write $Y = fY^0$ along $\varphi.w$. Then f is the solution of the equation

$$L_X(\log f) + \frac{1}{2}L_X(\log m) = 0,$$

which, with $f(w) = 1$, gives

$$f(\varphi^t(w)) = \left(\frac{m(w)}{m(\varphi^t(w))}\right)^{1/2}.$$

Finally,

$$Y(\varphi^t w) = \left(\frac{m(w)}{m(\varphi^t(w))}\right)^{1/2} Y^0(\varphi^t w). \tag{2.8}$$

Now, using (2.6), we have

$$h = H_X(Y) = -[X,Y] + D^X(Y) = -[X,Y]$$

along $\varphi.w$. Hence, from (2.8), we have

$$\begin{aligned} h = -[X,Y] &= -L_Y m\ X^0 - m\ [X^0, Y] \\ &= -L_Y m\ X^0 - m\ [X^0, \tfrac{m(w)}{m} Y^0] \\ &= -L_Y m\ X^0 - (m(w)m)^{1/2}\ [X^0, Y^0] + m(w)m\ L_{X^0}(m^{-1})\ Y^0 \\ &= -L_Y m\ X^0 + (m(w)m)^{1/2}\ h^0 - \tfrac{m(w)}{m}\ L_{X^0} m\ Y^0. \end{aligned}$$

□

2.1.7 Jacobi endomorphism and curvature

The Jacobi operator R^X is the C_X-linear operator defined by

$$R^X(X) = 0,\ R^X(Y) = p_v^X([X, H_X(Y)]),\ [R^X, H_X] = 0.$$

R^X is well defined thanks to lemma 2.1.1 and from lemma 2.1.5, we get that for any C_X vector field Z, $R^X(Z)$ is also a C_X vector field. Remark that R^X commutes with J^X. On VHM, we have

$$R^X = m^2 R^{X^0} + \big(\frac{1}{2} m L^2_{X^0} m - \frac{1}{4}(L_{X^0} m)^2\big) Id. \tag{2.9}$$

In the case X is the geodesic flow of a Riemannian metric g on M, the Jacobi operator allows to recover the curvature tensor R_g of g: for $u, v \in T_x M \smallsetminus \{0\}$, we have

$$R_g(u,v)u = \frac{d\pi(R^X V(x,[u]))}{\|u\|^2},$$

where $V(x,[u])$ is the unique vector in $\mathbb{R}.X(x,[u]) \oplus h^X HM(x,[u])$ such that $d\pi(V(x,[u])) = v$.

2.2 Dynamical formalism applied to Hilbert geometry

2.2.1 Construction

Let Ω be a strictly convex subset of $\mathbb{RP}^n$ with C^1 boundary. The geodesic flow φ^t of the Hilbert metric d_Ω is defined on the homogeneous tangent bundle $H\Omega = T\Omega \smallsetminus \{0\}/\mathbb{R}_+$: given a point x in

Ω and a direction $[\xi] \in H_x\Omega$, follow the line leaving x in the direction $[\xi]$ during the time t. Denote by X the generator of φ^t, that is, the second-order differential equation $X : H\Omega \longrightarrow TH\Omega$ such that

$$\frac{d}{dt}_{|t=0}\varphi^t = X.$$

Choose an affine chart and a Euclidean metric on it, such that Ω appears as a bounded subset of $\mathbb{R}^n$. Let $X^e : H\Omega \longrightarrow TH\Omega$ be the smooth second-order differential equation generating the Euclidean geodesic flow on $H\Omega$. Of course, this flow is not complete on $H\Omega$, that is, it is not defined for all $t \in \mathbb{R}$, but it is locally defined at least for small t. Since X and X^e have the same geodesics, we have $X = mX^e$ for some nonnegative function m, and we can check that

$$m(w) = 2\,\frac{|xx^+|\,|xx^-|}{|x^+x^-|},\ w = (x, [\xi]) \in H\Omega.$$

A direct computation gives that, for any $w = (x, [\xi]) \in H\Omega$,

$$L_{X^e}m\ (w) = 2\,\frac{|xx^+| - |xx^-|}{|x^+x^-|}\quad L^2_{X^e}m\ (w) = -\frac{4}{|x^+x^-|},\ L^n_{X^e}m = 0,\ n \geqslant 3,$$

so that $m \in C^1_{X^e}$. Thus the formalism introduced in the last section is relevant in this situation, X^e playing the role of X^0.

We immediately check that $R^{X^e} = 0$. Moreover, we have

Proposition 2.2.1. *Let Ω be a strictly convex subset of $\mathbb{RP}^n$ with C^1 boundary and X be the generator of the Hilbert metric on Ω. Then*

$$R^X|_{VH\Omega\oplus h^X H\Omega} = -Id|_{VH\Omega\oplus h^X H\Omega}.$$

This proposition means that, in some sense, such Hilbert geometries have constant strictly negative curvature.

Proof. We have

$$\begin{aligned}\frac{1}{2}mL^2_{X^e}m - \frac{1}{4}(L_{X^e}m)^2 \quad &= \frac{1}{2}.\ 2\,\frac{|xx^+||xx^-|}{|x^+x^-|}.\frac{-4}{|x^+x^-|} - \frac{1}{4}.\left(2\,\frac{|xx^+| - |xx^-|}{x^+x^-}\right)^2 \\ &= -\frac{4\ |xx^+||xx^-| + (|xx^+| - |xx^-|)^2}{|x^+x^-|^2} = -1.\end{aligned}$$

Using equation (2.9), we then get $R^X|_{VH\Omega\oplus h^X H\Omega} = -Id|_{VH\Omega\oplus h^X H\Omega}$. □

2.2.2 Hilbert's 1-form

The vertical derivative of a C^1 Finsler metric F on a manifold W is the 1-form on $TW \smallsetminus \{0\}$ defined for $Z \in T(TW \smallsetminus \{0\})$ by:

$$d_vF(x,\xi)(Z) = \lim_{\epsilon\to 0}\frac{F(x, \xi + \epsilon dp(Z)) - F(x,\xi)}{\epsilon},$$

where $p : TW \longrightarrow W$ is the bundle projection. This form depends only on the direction $[\xi]$: it is invariant under the Liouville flow generated by the Liouville field $D = \sum \xi_i \frac{\partial}{\partial \xi_i}$. As a consequence, $d_v F$ descends by homogeneity on HW to get a 1-form A called the Hilbert form of F.
Let X be the infinitesimal generator of the geodesic flow of F on HW. Since $[d\pi(X(x,[\xi]))] = [\xi]$, we can define A for any $Z \in THW$ by

$$A(Z) = \lim_{\epsilon \to 0} \frac{F(d\pi(X + \epsilon Z)) - 1}{\epsilon}.$$

Remark that $A(X) = 1$ and that $A(Y) = 0$ for any vertical vector field.

When X is smooth, the 2-form dA is well defined and we have

$$\imath_X dA = 0 \quad \ker A = VHW \oplus h^X HW.$$

The following proposition extends this result to our less regular Hilbert geometries.

Proposition 2.2.2. *Let Ω be a strictly convex subset of $\mathbb{RP}^n$ with C^1 boundary and A the Hilbert form of the Hilbert metric F on Ω. Then*

(i) $\ker A = VH\Omega \oplus h^X H\Omega$;

(ii) A is invariant under the geodesic flow of the Hilbert metric.

To prove the proposition, we have to make some computations on $H\Omega$, and to make them easier, we will use some special charts, that we introduce now. Choose a point $w = (x, [\xi]) \in H\Omega$ with orbit $\varphi.w$. *A chart adapted to this orbit* is an affine chart where the intersection $T_{x^+}\partial\Omega \cap T_{x^-}\partial\Omega$ is contained in the hyperplane at infinity, and a Euclidean structure on it so that the line (xx^+) is orthogonal to $T_{x^+}\partial\Omega$ and $T_{x^-}\partial\Omega$.

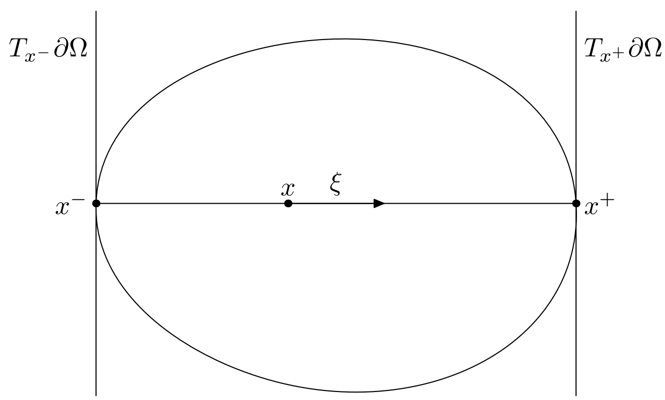

Figure 2.1: A good chart at $w = (x, [\xi])$

All along this work, when we talk about a **good chart** *at or a* **chart adapted** *to $w \in H\Omega$ or its orbit $\varphi.w$, we mean such a chart.*

In a good chart at w, we clearly have $L_Y m = 0$ along $\varphi.w$ for any $Y \in VH\Omega$.

Remark 2.2.3. *As a corollary of the following proof, we will get that, in a good chart at* $w = (x,[\xi]) \in H\Omega$,

$$d\pi(V_w H\Omega \oplus h_w^X H\Omega) = \left(\mathbf{x}\mathbf{x}^+\right)^\perp,$$

where orthogonality is taken with respect to the Euclidean metric of the chart. More generally, this implies that $d\pi(h_w^X H\Omega)$ *is the tangent space to the unit ball of* $F(x,.)$ *in the direction* $[\xi]$.

Proof of proposition 2.2.2. (i) We only have to prove that $h^X H\Omega \subset \ker A$. Let $w_0 = (x_0,[\xi_0])$ be any point in $H\Omega$ and fix a chart for Ω in $\mathbb{R}^n$ which is adapted to w_0, and where $x_0 = 0$ is the origin. Choose a small open neighbourhood U of w_0 in $H\Omega$. If U is small enough, we can choose coordinates $(x_1,\cdots,x_n,\xi_2,\cdots,\xi_n)$ on U such that:

- $w_0 = 0$ is the origin;
- for $w = (x,[\xi]) \in U$, the coordinates $(x_1,\cdots,x_n)$ of x are the Euclidean coordinates in $\mathbb{R}^n$ and $[\xi]$ is identified with the vector

$$\xi = \xi(w) = \frac{\partial}{\partial x_1} + \sum_{i=2}^{n} \xi_i \frac{\partial}{\partial x_i} \in T_x\Omega,$$

  where the ξ_i vary in a neighbourhood of 0. In other words, $[\xi] = [1:\xi_2:\cdots:\xi_n]$, where we make use of homogeneous coordinates on $H_x\mathbb{R}^n$.

We use the associated basis $\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial \xi_j}\right)_{1\leqslant i\leqslant n, 2\leqslant j\leqslant n}$ on the tangent space $TU \subset TH\Omega$. Remark that all along $\varphi.w_0 \cap U$, we have $\xi = \frac{\partial}{\partial x_1}$.

In this chart, we introduce a new second-order differential equation X^0 on U by

$$X^0(w) = X^0(x,[\xi]) = \frac{\partial}{\partial x_1} + \sum_{i=2}^{n} \xi_i \frac{\partial}{\partial x_i}.$$

In particular, we have $X^0(w) = \frac{\partial}{\partial x_1}$ along $\varphi.w_0 \cap U$, and $d\pi(X^0(x,[\xi])) = \xi$ on U. Moreover, X can be written as $X = kX^0$, where k is the C_X function defined on U by

$$k(w) = \frac{F(d\pi(X^0(w)))}{F(d\pi(X(w)))} = F(x,\xi(w)) = \frac{|\xi(w)|}{m(x,[\xi])},\ w = (x,[\xi]);$$

Along $\varphi.w_0$, we clearly have $L_Y k = 0$.

The vertical distribution on U is given by

$$VU = \mathrm{vect}\left\{\frac{\partial}{\partial \xi_i}\right\}_{i\in\{2,\cdots,n\}}.$$

Since $L_Y k = 0$ on $\varphi.w_0$, the pseudo-complex structure along $\varphi.w_0$ given by X^0 on $VU \oplus h^{X^0}U$ is very simple: we have

$$\forall j = 2,\cdots,n,\ \left[X^0,\frac{\partial}{\partial \xi_j}\right] = -\frac{\partial}{\partial x_j},\quad \left[X^0,\left[X^0,\frac{\partial}{\partial \xi_j}\right]\right] = 0,$$

hence

$$\forall j = 2, \cdots, n,\ v_{X^0}\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial \xi_j},\quad H_{X^0}\left(\frac{\partial}{\partial \xi_j}\right) = \frac{\partial}{\partial x_j},$$

so that

$$h^{X^0}U = \text{vect}\left\{\frac{\partial}{\partial x_i}\right\}_{i\in\{2,\cdots,n\}}. \tag{2.10}$$

Equation (2.3) can be applied with k instead of m. Any horizontal vector field $h \in h^X U$ along $\varphi.w_0$ can thus be written

$$h = kH_{X^0}(Y) + \frac{1}{2}(L_{X^0}k)Y, \tag{2.11}$$

for a certain vector field $Y \in VU$. Since $A(Y) = 0$, we have $A(h) = kA(H_{X^0}(Y))$; so, with (2.10) we only have to prove that for any $i \in \{2, \cdots, n\}$ and $w \in \varphi.w_0$, $A(w)(\frac{\partial}{\partial x_i}) = 0$. But

$$A\left(\frac{\partial}{\partial x_i}\right) = \lim_{\epsilon\to 0} \frac{F(d\pi(X + \epsilon\frac{\partial}{\partial x_i})) - 1}{\epsilon} = \lim_{\epsilon\to 0} \frac{F(d\pi(X^0 + \epsilon\frac{\partial}{\partial x_i})) - F(d\pi(X^0))}{\epsilon}$$

so that, for $w \in \varphi.w_0$,

$$A(w)\left(\frac{\partial}{\partial x_i}\right) = \lim_{\epsilon\to 0} \frac{F(x, \xi + \epsilon\frac{\partial}{\partial x_i})) - F(x,\xi)}{\epsilon} = D_{(x,\xi(w))}F\left(\frac{\partial}{\partial x_i}\right),$$

where we see F as a real valued function on $\Omega\times\mathbb{R}^n \subset \mathbb{R}^{2n}$ with coordinates $(x_1, \cdots, x_n, \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})$. Now, in our chart, from the formula giving F, we clearly have $\frac{\partial}{\partial x_i} \in \ker DF$ for any $i \in \{2, \cdots, n\}$, which proves that $h^X H\Omega \subset \ker A$ along $\varphi.w_0 \cap U$. This can be made for any point w_0, so that $h^X H\Omega \subset \ker A$ on $H\Omega$.

(ii) Since $A(X) = 1$, to prove that A is invariant under the flow, we only have to prove that its kernel is invariant, which from the first result is equivalent to proving that

$$p^X([X,Y]) = p^X([X,h]) = 0$$

for any vertical and horizontal vector fields Y and h.

- Since $[X,Y] = -H_X(Y) + D^X(Y)$, we clearly have $p^X([X,Y]) = 0$.

- Now let $w_0 \in H\Omega$ and consider the neighbourhood U of w_0 that we have considered before, with the same coordinates. Along $\varphi.w_0$, we have $p^X = p^{X^0}$, hence

  $$p^X([X,h]) = p^{X^0}(k[X^0,h] - L_h kX^0) = kp^{X^0}([X^0,h]) - L_h k.$$

  But, in our chart, we also have $L_h k = 0$ along $\varphi.w_0$: this can be seen directly or using equation (2.11). Then, if $h = H_X(Y)$ and $h^0 = H_{X^0}(Y)$, we have, with (2.11),

  $$p^{X^0}([X^0,h]) = p^{X^0}([X^0, kh^0 + \frac{1}{2}(L_{X^0}k)Y]) = kp^{X^0}([X^0,h^0]) = 0$$

  on $\varphi.w_0$.

Finally, $p^X([X,h]) = 0$ on $\varphi.w_0$, and thus on $H\Omega$. $\square$

2.3 Metrics on HM

Dynamical flows are usually studied on Riemannian manifolds, and most of the definitions or theorems are stated in this context: the definition of an Anosov system or a Lyapunov exponent, Oseledets' theorem...
Moreover, the manifold is often compact: in this case, all Riemannian metrics, and more generally all metrics defining the same topology, are equivalent; thus the choice of a specific metric is not important. When the manifold is not compact anymore, this choice has some importance: the behaviour of the flow has to be understood with respect to the chosen metric; it is not difficult to see that we can change a stable vector into an unstable one by changing the asymptotics of the metric.
In the case of geodesic flows on complete Riemannian open manifolds M, HM inherits a natural Riemannian metric from the base metric. In our case, we define a Finsler metric $\overline{F}$ on $H\Omega$, using the decomposition $TH\Omega = \mathbb{R}.X \oplus h^X H\Omega \oplus VH\Omega$: if $Z = aX + h + Y$ is some vector of $TH\Omega$, we set

$$\overline{F}(Z) = \left(|a|^2 + \frac{1}{2}\left((F(d\pi h))^2 + (F(d\pi J^X(Y)))^2\right)\right)^{1/2}. \tag{2.12}$$

Since the last decomposition is only continuous in general, $\overline{F}$ is also only continuous. It allows us to define the length of a C^1 curve $c : [0,1] \to H\Omega$ as

$$l(c) = \int_0^1 \overline{F}(\dot{c}(t))\ dt.$$

It induces a continuous metric $\overline{d}$ on $H\Omega$: the distance between two points $v, w \in H\Omega$ is the minimal length of a C^1 curve joining v and w.

Remark that, if $\Omega \subset \mathbb{RP}^2$, then $\overline{F}$ is actually a continuous Riemannian metric on $H\Omega$. In any case, it is obviously J^X-invariant on $h^X H\Omega \oplus VH\Omega$.

Most of the theorems in hyperbolic dynamics are stated on a Riemannian manifold. But the Riemannian metric is just a way of measuring length of vectors, so it is not a problem to work with a Finsler metric instead. However, as in the definition of Lyapunov regular points in the next chapter, determinants or angles are used, which are defined with respect to the Riemannian metric. This difficulty can be overpassed here by remarking that, using John's ellipsoid, it is always possible to define a Riemannian metric $\| \, . \, \|$ on $H\Omega$ which is bi-Lipschitz equivalent to $\overline{F}$: for any $Z \in TH\Omega$

$$\frac{1}{\sqrt{n}}\|Z\| \leqslant \overline{F}(Z) \leqslant \sqrt{n}\|Z\|,$$

where n is the dimension of Ω. Of course, there is no reason for this metric $\| \, . \, \|$ to be even continuous but it will not be important, as we will see later.

2.4 Stable and unstable manifolds

2.4.1 Parallel transport and action of the flow

We pick a tangent vector $Z(w) \in T_w H\Omega$. We want to study the behavior of the vector field $Z(\varphi^t(w)) = d\varphi^t(Z(w))$ defined along the orbit $\varphi.w$. Assume

$$Z(w) = Y(w) + h(w) \in V_w H\Omega \oplus h_w^X H\Omega.$$

Since $VH\Omega \oplus h^X H\Omega$ is invariant under the flow, we can write $Z = Y + h$. To find the expressions of Y and h, we write that, since Z is invariant under the flow, the Lie bracket $[X, Z]$ is 0 everywhere on $\varphi.w$.
For that, let $(h_1, \cdots, h_{n-1})$ be a basis of $h^X H\Omega$ of D^X-parallel vectors along $\varphi.w$, that is, $h_i^t = h_i(\varphi^t(w)) = T^t(h_i(w))$, where T^t denotes the parallel transport for D^X and $(h_i(w))_i$ is a basis of $h_w^X H\Omega$. Since D^X and v_X commute, the family $\{Y_i\} = \{v_X(h_i)\}$ is a basis of $VH\Omega$ of D^X-parallel vectors along $\varphi.w$. We immediately have $h_i = H_X(Y_i)$ and

$$[X, Y_i] = -h_i;\ [X, h_i] = -Y_i. \tag{2.13}$$

Indeed, since Y_i is parallel,

$$[X, Y_i] = D^X(Y_i) - H_X(Y_i) = -h_i.$$

To see the second equality, we write

$$[X, h_i] = p_h^X([X, h_i]) + p_v^X([X, h_i]) + p_X([X, h_i]).$$

But since h_i is parallel, we have

$$p_h^X([X, h_i]) = H_X \circ v_X([X, h_i]) = -H_X \circ v_X([X, [X, Y_i]]) = 2D^X(h_i) = 0,$$

and from the preceding proposition, $p_X([X, h_i]) = 0$; hence

$$[X, h_i] = p_v^X([X, h]) = p_v^X([X, H_X(Y_i)]) = R^X(Y_i) = -Y_i.$$

Then, in this basis, Z can be written as

$$Z = \sum a_i h_i + b_i Y_i,$$

where a_i and b_i are smooth real functions along $\varphi.w$, and the formulas (2.13) give

$$\begin{aligned}
[X, Z] = 0 \quad &\Longleftrightarrow \quad \sum_{i=1}^{n-1} (L_X a_i - b_i) h_i + (L_X b_i - a_i) Y_i = 0 \\
&\Longleftrightarrow \quad b_i = L_X a_i;\ a_i = L_X b_i,\ i = 1, \cdots, n-1 \\
&\Longleftrightarrow \quad b_i = L_X a_i;\ a_i = L_X^2 a_i,\ i = 1, \cdots, n-1.
\end{aligned}$$

From that we get the solution

$$Z(\varphi^t(w)) = d\varphi^t(Z(w)) = \sum A_i e^t (h_i^t + Y_i^t) + B_i e^{-t} (h_i^t - Y_i^t), \tag{2.14}$$

where

$$A_i = \frac{1}{2}(a_i(w) + b_i(w)),\ B_i = \frac{1}{2}(a_i(w) - b_i(w))$$

depend on the initial coordinates of Z at w.

It is then natural to define the distributions

$$E^u = \{Y + J^X(Y),\ Y \in VH\Omega\},\ E^s = \{Y - J^X(Y),\ Y \in VH\Omega\}.$$

Obviously, we have

Proposition 2.4.1. *E^u and E^s are invariant under the flow, and if $Z^s \in E^s,\ Z^u \in E^u$, then*

$$d\varphi^t(Z^u) = e^t T^t(Z^u),\ d\varphi^t(Z^s) = e^{-t}T^t(Z^s).$$

The operator J^X exhanges E^u and E^s and

$$d\varphi^t J^X(Z^s) = e^{2t} J^X(d\varphi^t Z^s).$$

Proof. The first equalities come directly from equations (2.14). For the second one, it is just the fact that J^X commutes with the parallel transport:

$$d\varphi^t J^X(Z^s) = e^t T^t J^X(Z^s) = e^t J^X T^t(Z^s) = e^{2t} J^X(d\varphi^t Z^s).$$

□

The decomposition $TH\Omega = \mathbb{R}.X \oplus E^s \oplus E^u$ will now be called the Anosov decomposition.

2.4.2 Stable and unstable manifolds

For $w = (x, [\xi]) \in H\Omega$, let us denote by $\mathcal{H}_w = \mathcal{H}_{x^+}(x)$ the horosphere based at $x^+ = \varphi^{+\infty}(w)$ and passing through x. The horosphere $\mathcal{H}_{\sigma w}$, where $\sigma : (x, [\xi]) \in H\Omega \longmapsto (x, [-\xi])$, is the horosphere $\mathcal{H}_{x^-}(x)$ the horosphere based at $x^- = \varphi^{-\infty}(w)$ and passing through x.

The stable and unstable manifolds at $w_0 = (x_0, [\xi_0]) \in H\Omega$ are the C^1 submanifolds of $H\Omega$ defined as

$$W^s(w_0) = \{w = (x, [xw_0^+]) \in H\Omega,\ x \in \mathcal{H}_w\},$$
$$W^u(w_0) = \{w = (x, [w_0^- x]) \in H\Omega,\ x \in H_{\sigma w}\}.$$

We can check (see [7]) that

$$W^s(w_0) = \{w \in H\Omega,\ \lim_{t\to+\infty} d_\Omega(\pi\varphi^t(w), \pi\varphi^t(w_0)) = 0\} = \{w \in H\Omega,\ \lim_{t\to+\infty} \overline{d}(\varphi^t(w), \varphi^t(w_0)) = 0\},$$

$$W^u(w_0) = \{w \in H\Omega,\ \lim_{t\to-\infty} d_\Omega(\pi\varphi^t(w), \pi\varphi^t(w_0)) = 0\} = \{w \in H\Omega,\ \lim_{t\to-\infty} \overline{d}(\varphi^t(w), \varphi^t(w_0)) = 0\}.$$

(Recall that $\pi : H\Omega \longrightarrow \Omega$ denotes the bundle projection)

Proposition 2.4.2. *The distributions E^u and E^s are the tangent spaces to W^s and W^u.*

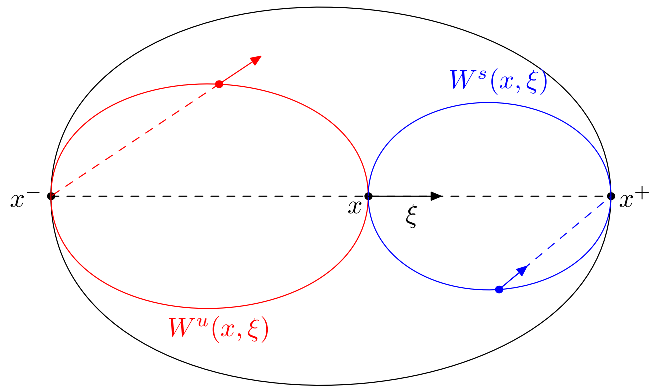

Figure 2.2: Stable and unstable manifolds

That will be a corollary of proposition 2.4.5. The image of a point $w = (x, \xi) \in H\Omega$ under the flow is denoted by $\varphi^t(w) = (x_t, \xi_t)$, for $t \in \mathbb{R}$. We first need a

Lemma 2.4.3. *We have*

$$\frac{|x_t x^-|}{|x_t x^+|} = e^{2t} \frac{|xx^-|}{|xx^+|}.$$

In particular the following asymptotic expansion holds:

$$|x_t x^+| = \frac{|xx^+|^2}{m(w)} e^{-2t} + O(e^{-4t}).$$

Proof. We have $d_\Omega(x, x_t) = t$, which implies

$$e^{2t} = \frac{|xx^-|}{|xx^+|} \frac{|x_t x^-|}{|x_t x^+|},$$

and yields the result. $\square$

Lemma 2.4.4. *In a good chart at $w = (x, [\xi])$ there exists a constant $C(w)$ such that, for any $Z(w) \in E^s(w) \cup E^u(w)$,*

$$\overline{F}(T^t Z(w)) = C(w)(|x_t x^+||x_t x^-|)^{1/2} \left(\frac{1}{|x_t y_t^+|} + \frac{1}{|x_t y_t^-|} \right),$$

where y_t^+ and y_t^- denote the points of intersection of the line $\{x + \lambda d\pi(Z(w))\}_{\lambda \in \mathbb{R}}$ with $\partial\Omega$ (see figure 2.3).

Proof. Assume for example that $Z(w) \in E^u(w)$. Then $Z(w) = h(w) + J^X(h(w))$, for some horizontal vector $h(w)$. Let h denote the parallel transport of $h(w)$, which is defined on the orbit $\varphi.w$. We have $T^t Z = h + J^X(h)$ on $\varphi.w$. In a good chart at w, lemma 2.1.6 gives

$$d\pi(h) = (m(w)m)^{1/2}\ d\pi(h^0);$$

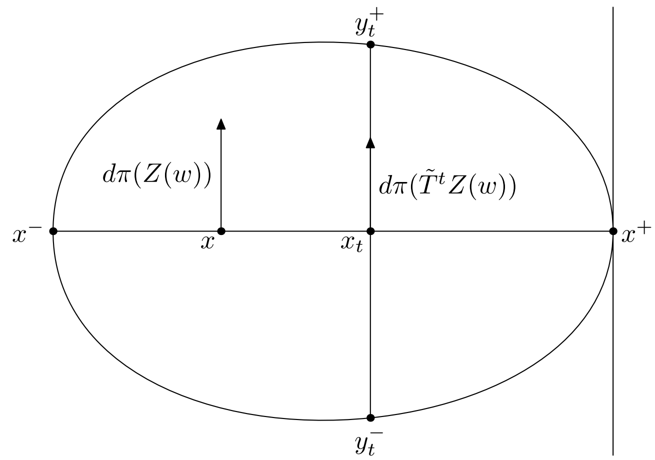

Figure 2.3: Parallel transport on $H\Omega$

in this case, since the chart is adapted, h^0 is just the Euclidean parallel transport of $h(w)$ along $\varphi.w$. In particular, $|d\pi(h^0)| = |d\pi(h^0(w))| = |d\pi(h(w))|$. Hence

$$\overline{F}(T^t Z(w)) = F(d\pi(h(\varphi^t w))) = \frac{|d\pi(h(w))|m(w)}{2} m(\varphi^t(w))^{1/2} \left(\frac{1}{|x_t y_t^+|} + \frac{1}{|x_t y_t^-|} \right).$$

□

Proposition 2.4.5. *Let $Z^s \in E^s$, $Z^u \in E^u$. Then $t \longmapsto \overline{F}(d\varphi^t Z^s)$ is a strictly decreasing bijection from $\mathbb{R}$ onto $(0, +\infty)$, and $t \longmapsto \overline{F}(d\varphi^t Z^u)$ is a strictly increasing bijection from $\mathbb{R}$ onto $(0, +\infty)$.*

Proof. Choose a stable vector $Z^s(w) \in E^s(w)$ and a chart adapted to $w = (x, [\xi])$. In that chart, the vector $d\pi(T^t Z^s(w))$ is orthogonal to $\mathbf{x_t x^+}$ with respect to the Euclidean structure on the chart; hence so are $\mathbf{x_t y_t^+}$ and $\mathbf{x_t y_t^-}$. We have from lemma 2.4.1,

$$\overline{F}(d\varphi^t Z^s(w)) = e^{-t} \overline{F}(T^t Z^s(w)).$$

Lemma 2.4.3 gives

$$|x_t x^-| = e^{2t} |x_t x^+| \frac{|x x^-|}{|x x^+|},$$

hence from lemma 2.4.4, there is a constant $C'(w)$ such that

$$\overline{F}(d\varphi^t Z^s(w)) = C'(w) \left(\frac{|x_t x^+|}{|x_t y_t^+|} + \frac{|x_t x^+|}{|x_t y_t^-|} \right)$$

The strict convexity of Ω implies that the function $h : t \mapsto \frac{|x_t x^+|}{|x_t y_t^+|} + \frac{|x_t x^+|}{|x_t y_t^-|}$ is strictly decreasing on $\mathbb{R}$, the C^1 regularity of $\partial\Omega$ that $\lim_{t \to +\infty} h(t) = 0$ and the strict convexity of Ω that $\lim_{t \to +\infty} h(t) = +\infty$.

The same computation holds for $t \mapsto \overline{F}(d\varphi^{-t}(Z^u))$ for $Z^u \in E^u$.

□

2.5 Uniform hyperbolicity of the geodesic flow

We want to consider now the geodesic flow φ^t of the Hilbert metric on the homogeneous bundle HM of a quotient manifold $M = \Omega/\Gamma$. The interesting part of the dynamics lies on the **nonwandering set** $N \subset HM$ of the flow. This is the closed φ^t-invariant set consisting of points $w \in HM$ such that for any neighbourhood $V \subset HM$ of w, the set of $t \in \mathbb{R}$ such that $\varphi^t(V) \cap V \neq \emptyset$ is unbounded, neither from above nor from below. Intuitively, this set consists of the points that come back close to their original positions infinitely often. We can thus expect some interesting dynamical properties on this set. On the contrary, a point $w \notin N$ will after some time leave forever to an unbounded part of the manifold.
Take for example a torus with a point removed, with a hyperbolic structure where the loop around the point is represented by a hyperbolic element γ. This manifold can be decomposed into a compact part, containing the "torus part" until the minimal geodesic c corresponding to γ, and an unbounded part, that we can picture as a trumpet, whose base would be the minimal geodesic c. Any geodesic entering the trumpet will never be able to come back into the compact part and will leave to infinity in the trumpet. The nonwandering set consists of these points that never enter the trumpet.

As can be expected, the nonwandering set is related to the limit set in the following way: if $\tilde{N} \subset H\Omega$ is the lift of N to $H\Omega$ under the covering map, then

$$\tilde{N} = \{w = (x, [\xi]),\ x^+, x^- \in \Lambda_\Gamma\}.$$

In particular, we have that $\tilde{N} \subset HC(\Lambda_\Gamma)$, that is, the projection of N on M is contained in the convex core of M.

All the things that were defined or proved on Ω in the last sections also make sense on the quotient manifold M, by using the covering map. We will keep using the same notations on M and HM since no confusion is possible. In particular, we still denote by X the second-order differential equation generating the geodesic flow φ^t on HM.

The next theorem is due to Yves Benoist in [7], but our study sheds a different light on the problem.

Theorem 2.5.1. *Assume $M = \Omega/\Gamma$ is compact. Then the geodesic flow φ^t is Anosov with invariant decomposition*

$$THM = \mathbb{R}.X \oplus E^s \oplus E^u;$$

that is, there exist constants $C, \alpha, \beta > 0$ such that for any $t \geqslant 0$,

$$\overline{F}(d\varphi^t(Z^s)) \leqslant Ce^{-\alpha t}\overline{F}(Z^s),\ Z^s \in E^s,$$

$$\overline{F}(d\varphi^{-t}(Z^u)) \leqslant Ce^{-\beta t}\overline{F}(Z^u),\ Z^u(w) \in E^u.$$

Proof. The argument we give here is Benoist's final argument in [7]. Choose $0 < a < 1$ and let

$$E^s_1 = \{Z^s \in E^s(w),\ \overline{F}(Z^s) = 1\} \subset THM.$$

From lemma 2.4.5, for any $Z^s \in E^s_1$, there is a unique time $T_a(Z^s)$ such that $\overline{F}(d\varphi^t(Z^s)) = a$. That defines a continuous function $T_a : E^s_1 \to \mathbb{R}$. Since E^s_1 is compact and $t \longmapsto \overline{F}(d\varphi^t Z^s)$ is strictly decreasing to 0, this function is bounded by some $t_a > 0$, such that

$$\forall t \geqslant t_a,\ \forall v \in E^s_1,\ \overline{F}(d\varphi^t(Z^s)) \leqslant a.$$

Thus we get, for t large enough and any $Z^s \in E^s_1(w)$,

$$\overline{F}(d\varphi^t(Z^s)) \leqslant a\overline{F}(d\varphi^{t-t_a}(Z^s)) \leqslant \cdots \leqslant a^{[t/t_a]}\overline{F}(d\varphi^{t-[t/t_a]t_a}(Z^s)) \leqslant M_a e^{-\alpha t},$$

with $M_a = \max\{\overline{F}(d\varphi^t(Z^s)),\ 0 \leqslant t \leqslant t_a, Z^s \in E^s_1\} < +\infty$ and $\alpha = -\log(a)/t_a > 0$.
That means that for any $Z^s \in E^s$,

$$\overline{F}(d\varphi^t(Z^s)) \leqslant C^2 M_a e^{-\alpha t}\overline{F}(Z^s).$$

□

In fact, one can prove that the same uniform property holds on the nonwandering set of a geometrically finite surface. That is the following

Theorem 2.5.2. *Let $M = \Omega/\Gamma$ be a geometrically finite surface. Then the geodesic flow φ^t is uniformly hyperbolic on the nonwandering set N with invariant decomposition*

$$THM = \mathbb{R}.X \oplus E^s \oplus E^u;$$

that is, there exist constants $C, \alpha, \beta > 0$ such that for any $t \geqslant 0$,

$$\overline{F}(d\varphi^t(Z^s)) \leqslant Ce^{-\alpha t}\overline{F}(Z^s),\ Z^s \in E^s,$$

$$\overline{F}(d\varphi^{-t}(Z^u)) \leqslant Ce^{-\beta t}\overline{F}(Z^u),\ Z^u \in E^u.$$

(There are some little mistakes in the proof of this result. See [27] for a proof without any mistake.)

In this case, we have to understand the behaviour of the flow in a cusp. But we know that far enough in the cusp, the geometry is almost hyperbolic, and we can hope the same for the flow. This hope is realized by lemma 2.5.3.

Let $\mathcal{P}$ be a maximal parabolic subgroup of Γ fixing $p \in \partial\Omega$. Recall proposition 1.3.5 and pick a small horoball H based at p such that there exist hyperbolic metrics $\mathtt{h}$ and $\mathtt{h}'$ on the quotient $\mathcal{C} = H/\mathcal{P}$ that satisfy

$$\frac{1}{C}\mathtt{h} \leqslant \mathtt{h}' \leqslant F \leqslant \mathtt{h} \leqslant C\mathtt{h}'$$

for some $C > 1$.

Lemma 2.5.3. *Let $w \in H\mathcal{C}$ and $t \geqslant 0$. If $\varphi^s(w) \in H\mathcal{C}$ for all $s \in [0,t]$ then, for any stable vector $Z(w) \in E^s(w)$, we have*

$$\overline{F}(d\varphi^t Z(w)) \leqslant Ke^{-t}\overline{F}(Z(w)),$$

for some constant $K > 0$.

Proof. We are going to compare the geodesic flows of F and $\mathtt{h}$ on $H\mathcal{C}$ to prove the proposition.

Let $X^{\mathtt{h}}$ be the generator of the geodesic flow $\varphi^t_{\mathtt{h}}$ of $\mathtt{h}$ on $H\mathcal{C}$. We have

$$F = g^{-1}\mathtt{h},\ X = gX^{\mathtt{h}},$$

for some C^1 function $g : H\mathcal{C} \longrightarrow [1, C]$.

The tangent space $TH\mathcal{C}$ can be decomposed in two ways, with respect to X or $X^{\mathtt{h}}$:

$$THM = \mathbb{R}.X \oplus VH\mathcal{C} \oplus h^X H\mathcal{C} = \mathbb{R}.X^{\mathtt{h}} \oplus VH\mathcal{C} \oplus h^{X^{\mathtt{h}}} H\mathcal{C}.$$

We have the endomorphism J^X on $VH\mathcal{C} \oplus h^X H\mathcal{C}$, that exhanges vertical and horizontal subspaces; the same for $J^{X^{\mathtt{h}}}$ on $VH\mathcal{C} \oplus h^{X^{\mathtt{h}}} H\mathcal{C}$. We define the two metrics $\overline{F}$ and $\overline{\mathtt{h}}$ on $H\mathcal{C}$ as in (2.12). $\overline{F}$ is a continuous Finsler metric and $\overline{\mathtt{h}}$ a smooth Riemannian metric, which is just the usual Sasaki metric.

Stable and unstable distributions of X and $X^{\mathtt{h}}$ are given by

$$E^u = \{Y + J^X(Y),\ Y \in VHM\},\ E^s = \{Y - J^X(Y),\ Y \in VHM\};$$

$$E^u_{\mathtt{h}} = \{Y + J^{X^{\mathtt{h}}}(Y),\ Y \in VHM\},\ E^s_{\mathtt{h}} = \{Y - J^{X^{\mathtt{h}}}(Y),\ Y \in VHM\}.$$

Let $Z(w) = h(w) - Y(w) \in E^s(w)$ be a X-stable vector, and denote by $Z = h - Y$ its parallel transport along $\varphi.w$. Then

$$d\varphi^t Z(w) = e^{-t} Z(\varphi^t w).$$

Hence

$$\overline{F}(d\varphi^t Z(w)) = e^{-t}\overline{F}(Z(\varphi^t(w)) = e^{-t} F(d\pi(h(\varphi^t w))),$$

so we just have to understand the behaviour of $F(d\pi(h))$ which is smaller than $\mathtt{h}(d\pi(h))$.

Denote by $Z^{\mathtt{h}}(w) = h^{\mathtt{h}}(w) - Y(w) \in E^s_{\mathtt{h}}(w)$ the $X^{\mathtt{h}}$-stable vector whose vertical part is the same than $Z(w)$, and by $Z^{\mathtt{h}} = h^{\mathtt{h}} - Y^{\mathtt{h}}$ its $X^{\mathtt{h}}$-parallel transport along $\varphi.w$. Lemma 2.1.6 gives

$$h = -L_Y g\ X^{\mathtt{h}} + (g(w)g)^{1/2}\ h^{\mathtt{h}} + g(w)g\ L_{X^{\mathtt{h}}}(g^{-1})\ Y^{\mathtt{h}}. \tag{2.15}$$

Lemma 2.5.4. *There exists $0 \leqslant \theta < \pi/2$ independant of w so that the angle (for $\mathtt{h}$) between $d\pi(h)$ and $d\pi(h^{\mathtt{h}})$ is smaller than θ.*

Proof. From remark 2.2.3, the space $d\pi_w(h^X HM)$ is nothing else than the tangent space to the unit ball of F in the direction $[\xi]$, if $w = (x, [\xi])$. The inequality $\dfrac{1}{C}\mathtt{h} \leqslant F \leqslant \mathtt{h}$ and the convexity of the balls allows us to conclude. (Just make a picture: the unit ball of $\mathtt{h}$ is a sphere and the unit ball of F is then between two spheres.) □

As a consequence of this lemma, we have

$$F(d\pi(h)) \leqslant \mathtt{h}(d\pi(h)) \leqslant \sqrt{1+\tan^2\theta}\ \mathtt{h}(\mathrm{proj}_{d\pi(h^{X^{\mathtt{h}}}HM)} d\pi(h)),$$

where $\mathrm{proj}_{d\pi(h^{X^{\mathtt{h}}}HM)}$ denotes the $\mathtt{h}$-orthogonal projection on $d\pi(h^{X^{\mathtt{h}}}HM)$. Equation (2.15) gives that

$$\mathrm{proj}_{d\pi(h^{X^{\mathtt{h}}}HM)} d\pi(h) = (g(w)g)^{1/2}\ d\pi(h^{\mathtt{h}}).$$

Thus, we get

$$F(d\pi(h)) \leqslant \sqrt{1+\tan^2\theta}\ (g(w)g)^{1/2}\mathtt{h}(d\pi(h^{\mathtt{h}})) \leqslant K\overline{\mathtt{h}}(Z^{\mathtt{h}}) = K\overline{\mathtt{h}}(Z^{\mathtt{h}}(w)),$$

where $K = C\sqrt{1+\tan^2\theta}$; the last equality comes from the fact that $Z^{\mathtt{h}}$ is parallel for $X^{\mathtt{h}}$ which is the flow of a Riemannian metric: the parallel transport is then an isometry.

We conclude, with the help of equation (2.15), that

$$\begin{aligned}
\overline{\mathtt{h}}(Z^{\mathtt{h}}(w)) = \mathtt{h}(d\pi(h^{\mathtt{h}}(w))) \quad &= g^{-1}(w)\mathtt{h}(d\pi(gh^{\mathtt{h}}(w))) \\
&= g^{-1}(w)\mathtt{h}(\mathrm{proj}_{d\pi(h^{X^{\mathtt{h}}}HM)}(d\pi(h(w)))) \\
&\leqslant g^{-1}(w)\mathtt{h}(d\pi(h(w))) \\
&= F(d\pi(h(w))) \\
&= \overline{F}(Z(w)).
\end{aligned}$$

Finally,

$$\overline{F}(d\varphi^t Z(w)) \leqslant K e^{-t}\overline{F}(Z(w)).$$

□

Proof of theorem 2.5.2. We can decompose N into a compact part N_0 and a finite number of parts $\mathcal{C}_i$, $1 \leqslant i \leqslant k$, corresponding to the cusps. In the case where there are no cusps, the proof in the compact case works. In the general case, from the proof in the compact case, we know there exist $0 < a \leqslant 1$ and $D > 0$, such that, for any $w \in N$ such that $\varphi^s(w) \in N_0$ for all $s \in [0,t]$, and any stable vector $Z(w) \in E^s(w)$,

$$\overline{F}(d\varphi^t Z(w)) \leqslant D e^{-at}\overline{F}(Z(w)).$$

Lemma 2.5.3 tells us that the cusps can be chosen so that there exists $K > 0$ such that, for any $w \in N$ such that $\varphi^s(w)$ stays in some $\mathcal{C}_i$ for all $s \in [0,t]$, and any stable vector $Z(w) \in E^s(w)$

$$\overline{F}(d\varphi^t Z(w)) \leqslant K e^{-t}\overline{F}(Z(w)).$$

Thus, for any $w \in N$ and any stable vector $Z(w) \in E^s(w)$, we have

$$\overline{F}(d\varphi^t Z(w)) \leqslant \max\{D, K\} e^{-at}\overline{F}(Z(w)).$$

□

Chapter 3

Lyapunov exponents

We study Lyapunov exponents of the geodesic flow. We see that the parallel transport contains all the information about them. Then we make a link between Lyapunov exponents, Oseledets' decomposition and the shape of the boundary. This link allows to define in a very simple way the Lyapunov manifolds, which are tangent to the spaces appearing in Oseledets' filtration.

3.1 Lyapunov regular points

Let $\varphi = (\varphi^t)$ be a C^1 flow on a Riemannian manifold $(W, \| \, . \, \|)$. We want to describe the behaviour of $d\varphi^t$ when t is large. For example, if W is compact and φ is an Anosov flow, then for any stable vector Z, the function $t \mapsto \|d\varphi^t Z\|$ is exponentially decreasing; on the contrary, if Z is an unstable vector, then it is exponentially increasing.

With this example in mind, the first idea is thus for a given general flow to look for some stable or unstable vectors, whose norm would decrease or increase exponentially fast. This behaviour is captured by looking at the limit

$$\lim_{t \to +\infty} \frac{1}{t} \log \|d\varphi^t(Z)\| = \chi(Z).$$

This limit does not exist in general but we can always look at the inferior and superior limits if needed. If it exists, then for any $\epsilon > 0$, there exists some $C_\epsilon > 0$ such that, whenever $t > 0$,

$$C_\epsilon^{-1} e^{(\chi(Z)-\epsilon)t} \leqslant \|d\varphi^t(Z)\| \leqslant C_\epsilon e^{(\chi(Z)+\epsilon)t}.$$

More generally, call $\overline{\chi}(Z)$ and $\underline{\chi}(Z)$ the superior and inferior limits. Then for any $\epsilon > 0$, there exists some $C_\epsilon > 0$ such that, whenever $t > 0$,

$$C_\epsilon^{-1} e^{(\underline{\chi}(Z)-\epsilon)t} \leqslant \|d\varphi^t(Z)\| \leqslant C_\epsilon e^{(\overline{\chi}(Z)+\epsilon)t}.$$

The numbers $\overline{\chi}(Z)$ and $\underline{\chi}(Z)$ are called the upper and lower forward Lyapunov exponents of Z. When $\underline{\chi}(Z) > 0$ or $\overline{\chi}(Z) < 0$, that means that $\|d\varphi^t Z\|$ has exponential behaviour.

Let us state clearly the definitions.

Definitions 3.1.1. *Let $\varphi = (\varphi^t)$ be a C^1 flow on a Riemannian manifold $(W, \| \,.\, \|)$. The forward and backward upper Lyapunov exponents $\overline{\chi}_+(Z)$ and $\overline{\chi}_-(Z)$ of a vector $Z \in TW$ are defined by*

$$\overline{\chi}_\pm(Z) = \limsup_{t \to \pm\infty} \frac{1}{t} \log \|d\varphi^t(Z)\|.$$

The forward and backward lower Lyapunov exponents $\underline{\chi}_+(Z)$ and $\underline{\chi}_-(Z)$ of a vector $Z \in TW$ are defined by

$$\underline{\chi}_\pm(Z) = \liminf_{t \to \pm\infty} \frac{1}{t} \log \|d\varphi^t(Z)\|.$$

It is not difficult to see that these numbers can take only a finite number of values when Z varies in T_wW for a given $w \in W$. Namely there exists a filtration

$$\{0\} = F_0(w) \subsetneq F_1(w) \subsetneq \cdots \subsetneq F_p(w) = T_wW$$

and real numbers

$$\overline{\chi}_{1,+}(w) < \cdots < \overline{\chi}_{p,+}(w),$$

such that, for any vector $Z_i \in F_i(w) \smallsetminus F_{i-1}(w)$, $1 \leqslant i \leqslant p$,

$$\limsup_{t \to +\infty} \frac{1}{t} \log \|d\varphi^t(Z_i)\| = \overline{\chi}_{i,+}(w).$$

The same occurs for backward and lower Lyapunov exponents.

We will be interested in the case where all these numbers coincide:

Definitions 3.1.2. *Let $\varphi = (\varphi^t)$ be a C^1 flow on a Riemannian manifold W. A point $w \in W$ is said to be* **regular** *if there exist a φ^t-invariant decomposition*

$$TW = E_1 \oplus \cdots \oplus E_p$$

along $\varphi.w$ and real numbers

$$\chi_1(w) < \cdots < \chi_p(w),$$

such that, for any vector $Z_i \in E_i \smallsetminus \{0\}$,

$$\lim_{t \to \pm\infty} \frac{1}{t} \log \|d\varphi^t(Z_i)\| = \chi_i(w), \tag{3.1}$$

and

$$\lim_{t \to \pm\infty} \frac{1}{t} \log |\det d\varphi^t| = \sum_{i=1}^{p} \dim E_i \ \chi_i(w). \tag{3.2}$$

The point w is said to be **forward** *or* **backward regular** *if this behaviour occurs only when t goes respectively to $+\infty$ or $-\infty$.*

The numbers $\chi_i(w)$ associated to a regular point w are called the Lyapunov exponents of the flow at w. Let $F_i^+ = \oplus_{k=1}^{i} E_i$ for $1 \leqslant i \leqslant p$. Then

$$\{0\} = F_0^+ \subsetneq F_1^+ \subsetneq \cdots \subsetneq F_p^+ = TW$$

is a φ^t-invariant filtration of TW along $\varphi.w$ such that, for any vector $Z_i \in F_i^+ \setminus F_{i-1}^+$, $1 \leqslant i \leqslant p$,

$$\lim_{t\to+\infty} \frac{1}{t} \log \|d\varphi^t(Z_i)\| = \chi_i.$$

Similarly, if $F_i^- = \oplus_{k=i}^p E_i$ for $1 \leqslant i \leqslant p$ then

$$TW = F_1^- \supsetneq \cdots \supsetneq F_p^- \supsetneq F_{p+1}^- = \{0\}$$

is a φ^t-invariant filtration of TW along $\varphi.w$ such that, for any vector $Z_i \in F_i^- \setminus F_{i+1}^-$, $1 \leqslant i \leqslant p$,

$$\lim_{t\to-\infty} \frac{1}{t} \log \|d\varphi^t(Z_i)\| = \chi_i.$$

We will call the decomposition

$$TW = E_1 \oplus \cdots \oplus E_p$$

and the filtrations

$$F_1^+ \subsetneq \cdots \subsetneq F_p^+,\ F_1^- \supsetneq \cdots \supsetneq F_p^-,$$

the Lyapunov or Oseledets decomposition and filtrations.

In our case, we do not have a smooth Riemannian metric on $H\Omega$ as in the last definition; instead, we have a (noncontinuous) Riemannian metric $\| \,.\, \|$ and a continuous Finsler metric $\overline{F}$ which are bi-Lipschitz equivalent. Then equation (3.1) will be replaced by

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \overline{F}(d\varphi^t(Z_i)) = \chi_i(w). \tag{3.3}$$

In equation (3.2), the quantity $|\det d\varphi^t|$ represents the effect of the flow on the Riemannian volume vol: if A is some Borel subset of $T_wW \simeq \mathbb{R}^n$ with non-zero volume, then

$$|\det d_w\varphi^t| = \frac{vol_{\varphi^t(w)}(d\varphi^t A)}{vol_w(A)}.$$

When we deal with the geodesic flow of some Riemannian manifold M, this volume is preserved provided we chose the usual Riemannian metric on HM, inherited from the basis, whose volume is just the Liouville measure. Here, $|\det d_w\varphi^t|$ has to be understood with respect to the (noncontinuous) Riemannian metric $\| \,.\, \|$ or as

$$vol_{\varphi^t(w)}(d\varphi^t B(w,1)),$$

where vol denotes the Busemann volume of $\overline{F}$ and $B(w,1)$ is the $\overline{F}$-unit ball in $T_wH\Omega$. We recall that the Busemann volume of $\overline{F}$ is the volume form such that $vol_w(B(w,1)) = 1$. In what follows, we will still use the notation det.

3.2 Lyapunov exponents in Hilbert geometry

3.2.1 Lyapunov exponents and Oseledets decomposition

A regular point $w \in H\Omega$ has always 0 as Lyapunov exponent since $\overline{F}(X) = 1$. We will say that w has no zero Lyapunov exponent if X is the only vector to have non exponential behaviour; that is, the subspace E_0 corresponding to the exponent 0 along $\varphi.w$ has dimension 1.

Proposition 2.4.5 implies that if w is a regular point, then $\chi(Z^s) \leqslant 0$ and $\chi(Z^u) \geqslant 0$ for any $Z^s \in E^s(w),\ Z^u \in E^u(w)$. Furthermore, if $Z^s \in E^s(w)$, then $Z^u = J^X Z^s \in E^u(w)$ and proposition 2.4.1 gives
$$\overline{F}(Z^s) = e^{-2t}\overline{F}(Z^u),$$
so that
$$\chi(Z^u) = 2 + \chi(Z^s).$$
Now, choose a tangent vector Z at a regular point w whose Lyapunov exponent is 0. Z can be written as $Z = aX + Z^u + Z^s$ for some $a \in \mathbb{R},\ Z^s \in E^s,\ Z^u \in E^u$. Since
$$\lim_{t\to\pm\infty} \frac{1}{t} \log \overline{F}(d\varphi^t(Z)) = 0,$$
we conclude that $\chi(Z^u) = \chi(Z^s) = 0$. Thus, the subspace E_0 corresponding to the exponent 0 can be decomposed as
$$E_0 = \mathbb{R}.X \oplus E^- \oplus E^+,$$
where $E^- \subset E^s,\ E^+ \subset E^u$.

At a regular point, the Oseledets decomposition can thus be written in the following way:
$$TH\Omega = E_0^s \oplus (\oplus_{i=1}^p E_i^s) \oplus E_{p+1}^s \oplus \mathbb{R}.X \oplus E_0^u \oplus (\oplus_{i=1}^p E_i^u) \oplus E_{p+1}^u, \tag{3.4}$$
with the relations
$$E_i^s = J^X(E_i^u),\ 0 \leqslant i \leqslant p.$$
The subspaces E_0^s and E_0^u, or E_{p+1}^s and E_{p+1}^s, might be $\{0\}$; w has no zero Lyapunov exponent if and only if all of them are actually $\{0\}$. The corresponding Lyapunov exponents are
$$-2 = \chi_0^- < \chi_1^- < \cdots < \chi_p^- < \chi_{p+1}^- = 0 = \chi_0^+ < \chi_1^+ < \cdots < \chi_p^+ < \chi_{p+1}^+ = 2,$$
with
$$\chi_i^+ = \chi_i^- + 2,\ 0 \leqslant i \leqslant p.$$
If w has no zero Lyapunov exponent then all the Lyapunov exponents at w are *strictly* between -2 and 2. That will be the case in most of our applications.

We can simplify a bit this exposition by going down to the base manifold Ω. Indeed, we see that some informations, namely those given by stable and unstable parts, are redundant and we can get rid of that.

Choose $Z_i^u \in E_i^u$ corresponding to the Lyapunov exponent χ_i^+. Then, from proposition 2.4.1, $d\varphi^t(Z_i^u) = e^t T^t(Z_i^u)$, hence

$$\chi_i^+ = \lim_{t\to\pm\infty} \frac{1}{t} \log \overline{F}(d\varphi^t(Z_i^u)) = 1 + \lim_{t\to\pm\infty} \frac{1}{t} \log \overline{F}(T^t(Z_i^u)).$$

For the corresponding stable vector $Z_i^s = J^X(Z_i^u)$, we have $d\varphi^t(Z_i^s) = e^{-t}T^t(Z_i^s)$ so that

$$\chi_i^- = -1 + \lim_{t\to\pm\infty} \frac{1}{t} \log \overline{F}(T^t(Z_i^s)) = -1 + \lim_{t\to\pm\infty} \frac{1}{t} \log \overline{F}(T^t(J^X(Z_i^u))) = -1 + \lim_{t\to\pm\infty} \frac{1}{t} \log \overline{F}(T^t(Z_i^u)),$$

because J^X commutes with T^t and $\overline{F}$ is J^X-invariant. The Lyapunov exponents of the parallel transport are defined as

$$\eta_i := \lim_{t\to\infty} \frac{1}{t} \log \overline{F}(T^t(Z_i^u)) = \lim_{t\to\infty} \frac{1}{t} \log \overline{F}(T^t(Z_i^s)),\ 0 \leqslant i \leqslant p+1,$$

and the corresponding Oseledets decomposition is given by

$$TH\Omega = \mathbb{R}.X \oplus \left(\oplus_{i=0}^{p+1}(E_i^s \oplus E_i^u)\right).$$

The Lyapunov exponents are then given by

$$\chi_i^+ = 1 + \eta_i,\ \chi_i^- = -1 + \eta_i. \tag{3.5}$$

3.2.2 Parallel transport on Ω

To eliminate the redundance of stable and unstable parts, we can define the parallel transport directly on Ω. Take a point $x \in \Omega$ and choose a geodesic $x(t) = \pi \circ \varphi^t(x, [\xi])$ leaving in the direction $[\xi]$. If $v \in T_x\Omega$, we define its parallel transport $T^t_{(x,[\xi])}v$ along this geodesic as $d\pi(T^t h(v))$ where $h(v)$ is the only vector in $\mathbb{R}.X(x,[\xi]) \oplus h^X_{(x,[\xi])}H\Omega$ such that $d\pi(h(v)) = v$.

Remark that, if $w = (x, [\xi]) \in H\Omega$ is regular, then

$$\oplus_{i=0}^{p+1}(E_i^s \oplus E_i^u) = E^s \oplus E^u = h^X H\Omega \oplus VH\Omega,$$

and the projection of this subspace on $T\Omega$ is $T_x\mathcal{H}_w$. We have

$$d\pi(T^t Z(w)) = T^t_w d\pi(Z(w)),$$

for any vector $Z(w) \in T_wH\Omega$. Furthermore,

$$d\pi(E_i^s \oplus E_i^u) = d\pi(E_i^s) = d\pi(E_i^u),$$

and the Oseledets decomposition at w thus induces a decomposition of $T_x\mathcal{H}_w$, which we call the Oseledets decomposition at x of the parallel transport along the geodesic $\varphi.w$, or in the direction $[\xi]$.

The parallel Lyapunov exponent of $v \in T_x\Omega$ along $\varphi.(x,[\xi])$, or in the direction $[\xi]$, if it exists, is given by

$$\eta((x,[\xi]), v) = \lim_{t\to+\infty} \frac{1}{t} \log F(T^t(v)).$$

These exponents are related to those of the parallel transport on $H\Omega$ by

$$\eta(w, Z(w)) = \eta(w, d\pi(Z(w))),\ \ Z(w) \in T_w H\Omega.$$

We can in the same way define upper, lower, forward and backward parallel Lyapunov exponents.

We then have the following description of regular points:

Proposition 3.2.1. *A point $w = (x, [\xi]) \in H\Omega$ is regular if and only if there exist a decomposition*

$$T_x\Omega = \mathbb{R}.\xi \oplus E_0(w) \oplus (\oplus_{i=1}^{p} E_i(w)) \oplus E_{p+1}(w),$$

with possibly $E_0(w) = \{0\}$ or $E_{p+1}(w) = \{0\}$, and numbers

$$-1 = \eta_0(w) < \eta_1(w) < \cdots < \eta_p(w) < \eta_{p+1}(w) = 1$$

such that, for any $v_i \in E_i \smallsetminus \{0\}$,

$$\lim_{t \to \pm\infty} \frac{1}{t} \log F(T_w^t(v_i)) = \eta_i(w),$$

and

$$\lim_{t \to \pm\infty} \frac{1}{t} \log |\det T_w^t| = \sum_{i=0}^{p+1} \eta_i(w) \dim E_i(w) := \eta(w).$$

Now if Z^s and Z^u are any stable and unstable vectors in $T_w H\Omega$, their Lyapunov exponents are given by

$$\chi(Z^s) = -1 + \eta(w, d\pi(Z^s)),\ \ \chi(Z^u) = 1 + \eta(w, d\pi(Z^u)).$$

Obviously, the same can be done on a quotient manifold $M = \Omega/\Gamma$, where we now have a parallel transport T^t along geodesics. This parallel transport is really different from the Riemannian one, even if they coincide when the metric is actually Riemannian. Here it is only possible to transport vectors along geodesics, and this transport is not an isometry for the Finsler metric F. In particular, if we transport a vector along a closed geodesic, then, after one loop, the transported vector will not necessarily coincide with the original one. This remark will be useful later in section 5.3.

3.2.3 The flip map

We already understood the symmetry that exists between stable and unstable distributions of the flow, which is a consequence of the fact it is a geodesic flow. We now investigate another symmetry, that exists thanks to the reversibility of the Finsler metric we are considering. The **flip map** is the C^∞ involutive diffeomorphism σ defined by

$$\begin{array}{cccc} \sigma : & H\Omega & \longrightarrow & H\Omega \\ & w = (x, [\xi]) & \longmapsto & (x, [-\xi]). \end{array}$$

The reversiblity of the metric implies that σ conjugates the flows φ^t and φ^{-t}:

$$\varphi^{-t} = \sigma \circ \varphi^t \circ \sigma.$$

We say that a subset A of $H\Omega$ is **symmetric** if it is σ-invariant, that is, $\sigma(A) = A$. A function $f : H\Omega \to \mathbb{R}$ is symmetric (resp. antisymmetric) if $f \circ \sigma = f$ (resp. $f \circ \sigma = -f$).

The main results about the flip map are summarized in the following lemma. The last point is the key argument for proving theorems 5.3.3 and 5.3.6.

Lemma 3.2.2. *Let Ω be a strictly convex proper open set $\Omega \subset \mathbb{RP}^n$ with C^1 boundary. Then*

(i) The differential $d\sigma$ anticommutes with J^X and preserves the decomposition $TH\Omega = \mathbb{R}.X \oplus h^X H\Omega \oplus VH\Omega$; σ is an $\overline{F}$-isometry and exchanges the stable and unstable foliations.

(ii) The set Λ of regular points is a symmetric set and $d\sigma$ preserves the Osedelets decomposition (3.4) by sending $E_i^s(w)$ to $E_i^u(\sigma(w))$, for any $w \in \Lambda$, $0 \leqslant i \leqslant p+1$.

(iii) The function $\eta : \Lambda \longrightarrow \mathbb{R}$ is antisymmetric.

Proof. (i) Clearly, $d\sigma(X) = -X$ and $d\sigma$ preserves $VH\Omega$. Now, just recall how v_X is defined: for any $Y \in VH\Omega$, we have $v_X(X) = v_X(Y) = 0$, and $v_X([X,Y]) = -Y$, so

$$d\sigma v_X(X) = v_X(d\sigma(X)) = 0 = d\sigma v_X(Y) = v_X(d\sigma(Y)),$$

and

$$v_X d\sigma([X,Y]) = v_X([d\sigma(X), d\sigma(Y)]) = v_X([-X, d\sigma(Y)] = d\sigma(Y) = -d\sigma v_X([X,Y]).$$

So $d\sigma \circ v_X = -v_X \circ d\sigma$. As for H_X (see section 2.1.4):

$$\begin{aligned} d\sigma H_X(Y) = d\sigma(-[X,Y] - \frac{1}{2} v_X[X,[X,Y]]) &= -[d\sigma(X), d\sigma(Y)] + \frac{1}{2} v_X[d\sigma(X), [d\sigma(X), d\sigma(Y)]] \\ &= [X, d\sigma(Y)] + \frac{1}{2} v_X[X,[X,d\sigma(Y)]] \\ &= -H_X(d\sigma(Y)). \end{aligned}$$

Finally, we get that $d\sigma$ and J^X anticommute, which implies the σ-invariance of $\overline{F}$. It also gives that, if $Z = Y + J^X(Y) \in E^u$, then $d\sigma(Z) = d\sigma(Y) - J^X d\sigma(Y) \in E^s$, hence $d\sigma(E^u) = E^s$, and conversely; so σ exchanges stable and unstable foliations.

(ii) If $w \in \Lambda$, then from the very definition 3.1.2 of a regular point,

$$\lim_{t\to+\infty} \frac{1}{t} \log \overline{F}(d_w\varphi^{-t}(Z)) = -\lim_{t\to+\infty} \frac{1}{t} \log \overline{F}(d_w\varphi^{t}(Z)) = -\chi(w,Z),$$

for $Z \in T_w H\Omega$. Since $\varphi^{-t} = \sigma \circ \varphi^t \circ \sigma$, we thus have

$$-\chi(w,Z) = \lim_{t\to+\infty} \frac{1}{t} \log \overline{F}(d_w\varphi^{-t}(Z)) = \lim_{t\to+\infty} \frac{1}{t} \log \overline{F}(d_{\sigma(w)}\varphi^{t}(d_w\sigma(Z))) = \chi(\sigma(w), d_w\sigma(Z)),$$

which proves that $\sigma(w)$ is also regular, hence Λ is symmetric. We also get the decomposition

$$T_{\sigma(w)} H\Omega = R.X(\sigma(w)) \oplus \left(\bigoplus_{i=0}^{p+1} (E_i^s(\sigma(w)) \oplus E_i^u(\sigma(w))) \right)$$

with

$$E_i^s(\sigma(w)) = d\sigma(E_i^u(w)),\ E_i^u(\sigma(w)) = d\sigma(E_i^s(w)).$$

(iii) We then have

$$\chi_i^+(\sigma(w)) = -\chi_{p+1-i}^-(w), \tag{3.6}$$

that is (recall (3.5)),

$$1+\eta_i(\sigma(w)) = -(-1+\eta_{p+1-i}(w)).$$

This implies

$$\eta_i(\sigma(w)) = -\eta_{p+1-i}(w),$$

and

$$\eta(\sigma(w)) = \sum_{i=1}^{p} \dim E_i(\sigma(w))\ \eta_i(\sigma(w)) = -\eta(w).$$

□

3.3 Oseledets' theorem

The essential result about regular points is the following theorem of Oseledets:

Theorem 3.3.1 (Osedelets' ergodic multiplicative theorem, [60]). *Let $\varphi = (\varphi^t)$ be a C^1 flow on a Riemannian manifold $(W, \| \,.\, \|)$ and μ a φ^t-invariant probability measure. If*

$$\frac{d}{dt}|_{t=0} \log \|d\varphi^{\pm t}\| \in L^1(W,\mu), \tag{3.7}$$

then the set Λ of regular points has full measure.

Assumption (3.7) means that the flow does not expand or contract locally too fast. This essentially allows us to use Birkhoff's ergodic theorem to prove the theorem.
This condition is always satisfied on a compact manifold, since the functions in (3.7) are actually bounded. Thus, on a compact manifold, the set of regular points has full measure for any invariant probability measure.

If m is a finite measure on a nonnecessarily compact manifold, then it is sufficient to prove such a condition of boundedness. That is what is done by the next lemma for our geodesic flow. Remark that in this case, we do not have C^1 metrics, so condition (3.7) has to be replaced by

$$\limsup_{t\to 0} \frac{1}{t}\log\|d\varphi^t\|,\ \liminf_{t\to 0}\frac{1}{t}\log\|d\varphi^t\| \in L^1(W,\mu).$$

Lemma 3.3.2. *Let $\Omega \subset \mathbb{RP}^n$ be a strictly convex proper open set with C^1 boundary. For any $Z^s \in E^s,\ Z^u \in E^u$,*

$$-2 \leqslant \liminf_{t\to 0}\frac{1}{t}\log \overline{F}(d\varphi^t Z^s) \leqslant \limsup_{t\to 0}\frac{1}{t}\log\overline{F}(d\varphi^t Z^s) \leqslant 0$$

and

$$0 \leqslant \liminf_{t\to 0}\frac{1}{t}\log \overline{F}(d\varphi^t Z^u) \leqslant \limsup_{t\to 0}\frac{1}{t}\log\overline{F}(d\varphi^t Z^u) \leqslant 2.$$

In particular, for any $t \in \mathbb{R}$ and $Z \in TH\Omega$,

$$e^{-2|t|}\overline{F}(Z) \leqslant \overline{F}(d\varphi^t(Z)) \leqslant e^{2|t|}\overline{F}(Z).$$

This lemma clearly implies the already known fact (coming from proposition 2.4.5) that Lyapunov exponents at a regular point are all between -2 and 2. But it is more precise: it gives that the rate of expansion/contraction is *at any time* between -2 and 2, not only asymptotically, and that is what is essential to apply Oseledets' theorem.

Proof. It is a direct corollary of proposition 2.4.5: we know that $t \mapsto \overline{F}(d\varphi^t Z^s)$ is decreasing, hence

$$\limsup_{t\to 0} \frac{1}{t} \log \overline{F}(d\varphi^t Z^s) \leqslant 0.$$

But we also know from proposition 2.4.1 that

$$\overline{F}(d\varphi^t Z^s) = e^{-2t}\overline{F}(d\varphi^t J^X(Z^s)).$$

Since $J^X(Z^s) \in E^u$, proposition 2.4.5 tells us that $t \mapsto \overline{F}(d\varphi^t J^X(Z^s))$ is increasing, hence

$$\liminf_{t\to 0} \frac{1}{t} \log \overline{F}(d\varphi^t J^X(Z^s)) \geqslant 0$$

and

$$\liminf_{t\to 0} \frac{1}{t} \log \overline{F}(d\varphi^t Z^s) \geqslant -2.$$

Using J^X, we get the second inequality, and by integrating, we get the last one. □

3.4 Lyapunov structure of the boundary

In this part, we give a link between Lyapunov exponents and the shape of the boundary at the endpoint of a regular orbit.

3.4.1 Motivation

We first give the idea in dimension 2. Let $w \in \Omega$ be a regular point and choose a vector v tangent to $\mathcal{H}_w$, with parallel Lyapunov exponent η. In a good chart at w, lemma 2.4.4 gives

$$F(T^t v) = C(w)(|x_t x^+||x_t x^-|)^{1/2}\left(\frac{1}{|x_t y_t^+|} + \frac{1}{|x_t y_t^-|}\right).$$

Assume that $|x_t y_t^-| \asymp |x_t y_t^+|$. Then

$$\lim_{t\to+\infty} \frac{1}{t} \log \frac{F(T^t v)}{|x_t x^+|^{1/2}} = -\lim_{t\to+\infty} \frac{1}{t} \log |x_t y_t^+|,$$

hence, dividing by $\log |x_t x^+|^{1/2}$,

$$\lim_{t\to+\infty} \frac{\log F(T^t)}{\log |x_t x^+|^{1/2}} - 1 = -\lim_{t\to+\infty} \frac{\log |x_t y_t^+|}{\log |x_t x^+|^{1/2}}.$$

Since $|x_t x^+| \asymp e^{-2t}$, that yields

$$\lim_{t\to+\infty} \frac{\log|x_t y_t^+|}{\log|x_t x^+|} = \frac{1+\eta}{2}.$$

Let $f : T_{x^+}\partial\Omega \longrightarrow \mathbb{R}^n$ be the graph of $\partial\Omega$ at x^+, so that $|x_t x^+| = f(|x_t y_t^+|)$. We thus obtain

$$\lim_{s\to 0} \frac{\log f(s)}{\log s} = \frac{2}{1+\eta},$$

that is, for any $\epsilon > 0$, there exists $C > 0$ such that

$$C^{-1} s^{\frac{2}{1+\eta}+\epsilon} \leqslant f(s) \leqslant C s^{\frac{2}{1+\eta}-\epsilon}. \tag{3.8}$$

This link was first established in [25] for divisible convex sets, where the condition $|x_t y_t^-| \asymp |x_t y_t^+|$ is always satisfied. In order to generalize it, we must introduce new definitions. It may be a bit fastidious so you could prefer going directly to proposition 3.4.9, and have a look to the part in between when it is needed.

3.4.2 Locally convex submanifolds of $\mathbb{RP}^n$

Definition 3.4.1. *A codimension 1 C^0 submanifold N of $\mathbb{R}^n$ is* **locally (strictly) convex** *if for any $x \in N$, there is a neighbourhood V_x of x in $\mathbb{R}^n$ such that $V_x \smallsetminus N$ consists of two connected components, one of them being (strictly) convex.*

A codimension 1 C^0 submanifold N of $\mathbb{RP}^n$ is **locally** *(strictly)* **convex** *if its trace in any affine chart is locally (strictly) convex.*

Obviously, to check if $N \subset \mathbb{RP}^n$ is convex around x, it is enough to look at the trace of N in *one* affine chart at x. Choose a point $x \in N$ in a locally convex submanifold N and an affine chart centered at x. We can find a tangent space T_x of N at x such that $V_x \cap N$ is entirely contained in one of the closed half-spaces defined by T_x. We can then endow the chart with a suitable Euclidean structure, so that, around x, N appears as the graph of a convex function $f : U \subset T_x \longrightarrow [0, +\infty)$ defined on an open neighbourhood U of $0 \in T_x$. This function is (at least) as regular as N, is positive, $f(0) = 0$ and $f'(0) = 0$ if N is C^1 at x. When N is strictly locally convex, then f is strictly convex, in particular $f(v) > 0$ for $v \neq 0$.

In what follows, we are interested in the shape of the boundary $\partial\Omega$ of Ω at some specific point, or, more generally, in the local shape of locally strictly convex C^1 submanifolds of $\mathbb{RP}^n$. Denote by $\texttt{Cvx}(n)$ the set of strictly convex C^1 functions $f : B = B(0,1) \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $f(0) = f'(0) = 0$, where B denotes the closed unit ball in $\mathbb{R}^n$. We look for properties of such functions which are invariant by projective transformations.

3.4.3 Approximate α-regularity

We introduce here the main notion of approximate α-regularity, describe its meaning and prove some useful lemmas.

Definitions 3.4.2. *A function $f \in \mathtt{Cvx}(1)$ is said to be* **approximately α-regular**, $\alpha \in [1, +\infty]$, *if*

$$\lim_{t \to 0} \frac{\log \dfrac{f(t) + f(-t)}{2}}{\log |t|} = \alpha.$$

This property is clearly invariant by affine transformations, and in particular by change of Euclidean structure. It is in fact invariant by projective ones, but we do not need to prove it directly, since it will be a consequence of proposition 3.4.9.
Obviously, the function $t \in \mathbb{R} \mapsto |t|^\alpha$, $\alpha > 1$ is approximately α-regular. To be α-regular, with $1 < \alpha < +\infty$, means that we roughly behave like $t \mapsto |t|^\alpha$.
The case $\alpha = \infty$ is a particular one: f is ∞-regular means that for any $\alpha \geqslant 1$, $f(t) \ll |t|^\alpha$ for small $|t|$. An easy example of such a function is provided by $f : t \longmapsto e^{-1/t^2}$. On the other side, f is 1-regular means that for any $\alpha > 1$, $f(t) \gg |t|^\alpha$. An example of function which is 1-regular is provided by the Legendre transform of the last one.

In the case where $1 < \alpha < +\infty$, we can state the following equivalent definitions. The proof is straightforward.

Lemma 3.4.3. *Let $f \in \mathtt{Cvx}(1)$ and $1 < \alpha < +\infty$. The following propositions are equivalent:*

- *f is approximately α-regular;*
- *for any $\epsilon > 0$ and small $|t|$,*
$$|t|^{\alpha+\epsilon} \leqslant \frac{f(t) + f(-t)}{2} \leqslant |t|^{\alpha-\epsilon};$$
- *the function $t \longmapsto \dfrac{f(t) + f(-t)}{2}$ is $C^{\alpha-\epsilon}$ and $\alpha + \epsilon$-convex at 0 for any $\epsilon > 0$.*

To understand the last proposition, we recall the following

Definitions 3.4.4. *Let $\alpha, \beta \geqslant 1$ We say that a function $f \in \mathtt{Cvx}(n)$ is*

- *C^α if for small $|t|$, $t \in \mathbb{R}^n$, there is some $C > 0$ such that*
$$f(t) \leqslant C|t|^\alpha;$$
- *β-convex if for small $|t|$, $t \in \mathbb{R}^n$, there is some $C > 0$ such that*
$$f(t) \geqslant C|t|^\beta.$$

We now give another equivalent definition of approximate regularity, that shows the relation with the motivation above. Proposition 3.4.9 is the most important consequence of it.

Let $f \in \mathtt{Cvx}(1)$. Denote by $f^+ = f^{-1}_{|_{[0,1]}}$ and $f^- = -f^{-1}_{|_{[-1,0]}}$. These functions are both nonnegative, increasing and concave and their value at 0 is 0; they are C^1 on $(0, 1]$ and their tangent at 0 is vertical.

The harmonic mean of two numbers $a, b > 0$ is defined as

$$H(a,b) = \frac{2}{a^{-1} + b^{-1}}.$$

The harmonic mean of two functions $f,\ g : X \to (0, +\infty)$ defined on the same set X is the function $H(f,g)$ defined for $x \in X$ by

$$H(f,g)(x) = H(f(x), g(x)) = \frac{2}{\frac{1}{f(x)} + \frac{1}{g(x)}}.$$

Proposition 3.4.5. *A function $f \in \mathtt{Cvx}(1)$ is approximately α-regular, $\alpha \in [1, +\infty]$ if and only if*

$$\lim_{t \to 0^+} \frac{\log H(f^+, f^-)(t)}{\log t} = \alpha^{-1},$$

with the convention that $\frac{1}{+\infty} = 0$.

Proof. As we will see, it is enough to take f continuous, so by replacing f^+ and f^- by $\min(f^+, f^-)$ and $\max(f^+, f^-)$, we can assume that $f^+ \leqslant f^-$, that is $f(t) \geqslant f(-t)$ for $t \geqslant 0$. Now, assuming that the limit exists,

$$\lim_{t \to 0^+} \frac{\log H(f^+, f^-)(t)}{\log t} = -\lim_{t \to 0^+} \frac{\log \left(\frac{1}{f^+(t)} + \frac{1}{f^-(t)} \right)}{\log t} = \lim_{t \to 0^+} \frac{\log f^+(t)}{\log t} - \lim_{t \to 0^+} \frac{\log \left(1 + \frac{f^+(t)}{f^-(t)} \right)}{\log t}.$$

Since $f^+ \leqslant f^-$, the second limit is 0, and the first one is

$$\lim_{t \to 0^+} \frac{\log f^+(t)}{\log t} = \lim_{u \to 0^+} \frac{\log u}{\log f(u)}.$$

But, since $f(u) \geqslant f(-u)$ for $u \geqslant 0$, we get

$$\lim_{u \to 0^+} \frac{\log u}{\log \frac{f(u)+f(-u)}{2}} = \lim_{u \to 0^+} \frac{\log u}{\log f(u) + \log \left(1 + \frac{f(-u)}{f(u)} \right)} = \lim_{u \to 0^+} \frac{\log u}{\log f(u)},$$

hence the result. $\square$

The last construction can be generalized in a way that will be useful later, for proving proposition 3.4.9. Let $f \in \mathtt{Cvx}(1)$ and pick $a > 0$. We define two new "inverse functions" $f_a^+(s)$ and $f_a^-(s)$ for $s \in [0, \epsilon]$, $\epsilon > 0$ small enough, depending on a; these are positive functions defined by the equations

$$f(f_a^+(s)) = s - s f_a^+(s); f(-f_a^-(s)) = s + s f_a^-(s).$$

Geometrically, for $s \in [0, \epsilon]$ on the vertical axis, the line (as) cuts the graph of f at two points a^+ and $a-$, with s between a^+ and a^-; $f_a^+(s)$ and $f_a^-(s)$ are the abscissae of a^+ and a^- (c.f. figure 3.4.3). f^+ and f^- can be considered as $f_{+\infty}^+$ and $f_{+\infty}^-$.

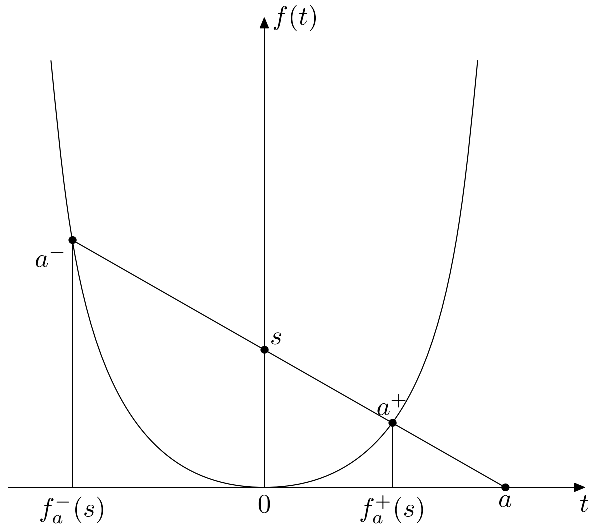

Figure 3.1: Construction of new inverses

Lemma 3.4.6. *Let $f \in \mathtt{Cvx}(1)$ and $a > 0$. The functions $\dfrac{f_a^+}{f^+}$ and $\dfrac{f_a^-}{f^-}$ can be extended by continuity at 0 by*

$$\frac{f_a^+}{f^+}(0) = \frac{f_a^-}{f^-}(0) = 1.$$

In particular, for $s > 0$ small enough,

$$f^+(s) \asymp f_a^+(s),\;\; f^-(s) \asymp f_a^-(s).$$

Proof. We prove it for f^+ and f_a^+. Clearly, we have $\frac{f_a^+(s)}{f^+(s)} \leqslant 1$. Since f is convex and $f(0) = 0$, we get

$$s - sf_a^+(s) = f(f_a^+(s)) = f\left(\frac{f_a^+(s)}{f^+(s)} f^+(s)\right) \leqslant \frac{f_a^+(s)}{f^+(s)} f(f^+(s)) = \frac{f_a^+(s)}{f^+(s)} s.$$

Hence, for $0 < s \leqslant \epsilon < 1$

$$\frac{f_a^+(s)}{f^+(s)} \geqslant 1 - f_a^+(s) \geqslant 1 - f_a^+(\epsilon).$$

The function $\dfrac{f_a^+}{f^+}$ can even be extended at 0 by $\dfrac{f_a^+}{f^+}(0) = 1$ ◻

The result to remember is the following consequence of lemmas 3.4.6 and 3.4.5:

Corollary 3.4.7. *Pick $a > 0$. A function $f \in \mathtt{Cvx}(1)$ is approximately α-regular if and only if*

$$\lim_{t \to 0^+} \frac{\log H(f_a^+, f_a^-)(t)}{\log t} = \alpha^{-1}.$$

We end this section by extending the definitions in higher dimensions:

Definitions 3.4.8. *A function $f \in \mathtt{Cvx}(n)$ is said to be* **Lyapunov-regular** *at x if it is approximately regular in any direction, that is, for any $v \in \mathbb{R}^n \smallsetminus \{0\}$, there exists $\alpha(v) \in [1, \infty]$ such that*

$$\lim_{t\to 0} \frac{\log \dfrac{f(tv)+f(-tv)}{2}}{\log |t|} = \alpha(v).$$

Let $f \in \mathtt{Cvx}(n)$. The upper and lower Lyapunov exponents $\overline{\alpha}(v)$ and $\underline{\alpha}(v)$ of $v \in \mathbb{R}^n$ are defined by

$$\overline{\alpha}(v) = \limsup_{t\to 0} \frac{\log \dfrac{f(tv)+f(-tv)}{2}}{\log |t|},$$

$$\underline{\alpha}(v) = \liminf_{t\to 0} \frac{\log \dfrac{f(tv)+f(-tv)}{2}}{\log |t|}.$$

The function is then Lyapunov-regular if and only if the preceding limits are indeed limits in $[1, +\infty]$, that is, for any $v \in \mathbb{R}^n$, $\overline{\alpha}(v) = \underline{\alpha}(v)$. Obviously, lemma 3.4.5 and corollary 3.4.7 have their counterpart in higher dimensions.

3.4.4 Lyapunov-regularity of the boundary

If Ω is a bounded convex set in the Euclidean space $\mathbb{R}^n$ with C^1 boundary, the graph of $\partial\Omega$ at x is the function

$$\begin{array}{rccl} f: & U \subset T_x\partial\Omega & \longrightarrow & \mathbb{R}^n \\ & u & \longmapsto & \{u + \lambda n(x)\}_{\lambda\in\mathbb{R}} \cap \partial\Omega, \end{array}$$

where $n(x)$ denotes a normal vector to $\partial\Omega$ at x, and U is a sufficiently small open neighbourhood of $x \in \partial\Omega$ for the function to be defined.
The following innocent-like proposition, whose proof is now straightforward, allows us to understand a lot about the asymptotic dynamics of the flow. Also, it gives an important tool for intuition.

Proposition 3.4.9. *Let Ω be a strictly convex proper open set of $\mathbb{RP}^n$ with C^1 boundary. Pick $x^+ \in \partial\Omega$, choose any affine chart containing x^+ and a Euclidean metric on it.*
Then for any $v \in T_{x^+}\partial\Omega$, any $w \in H\Omega$ ending at x^+, we have

$$\overline{\eta}_+(w, v(w)) = \frac{2}{\underline{\alpha}(x^+, v)} - 1,\ \underline{\eta}_+(w, v(w)) = \frac{2}{\overline{\alpha}(x^+, v)} - 1,$$

where $v(w)$ is any vector in $T_x\mathcal{H}_w \cap (\mathbb{R}.v \oplus \mathbb{R}.\xi) \subset \mathbb{R}^n$ and $\underline{\alpha}(x^+, v)$ and $\overline{\alpha}(x^+, v)$ are the lower and upper Lyapunov exponents of $\partial\Omega$ at x^+ in the direction v, as defined at the very end of the last section.

Proof. Let $w = (x, [\xi])$ be a point ending at x^+, and $(x_t, [\xi_t]) = \varphi^t(x, [\xi])$ its image by φ^t. The vector $T^t v(w)$ is at any time contained in the plane generated by ξ and v, thus, by working in restriction to this plane, we can assume that $n = 2$.
We cannot choose a good chart at w, since the chart is already fixed. But, by affine invariance,

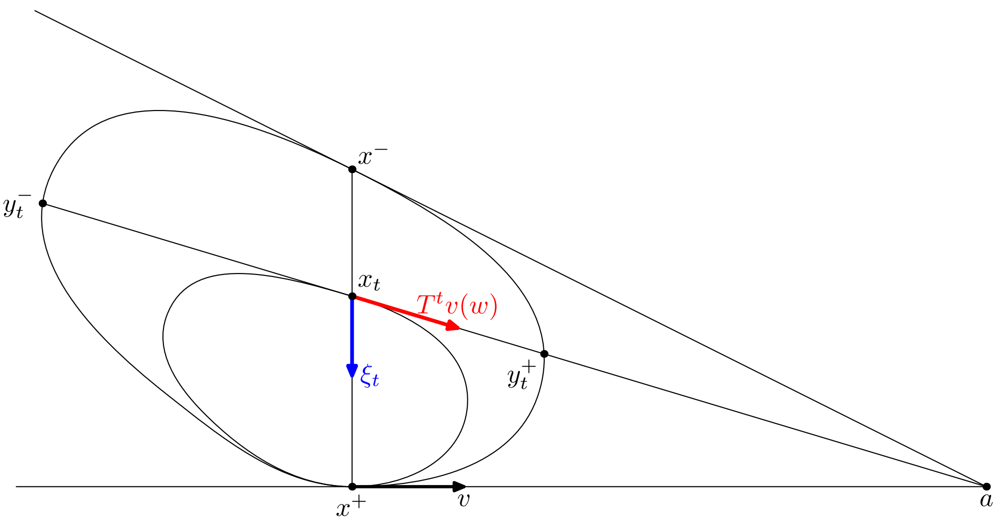

Figure 3.2: For proposition 3.4.9

we can choose the Euclidean metric $| \, . \, |$ and ξ_t so that $\xi \perp T_{x^+}\partial\Omega = \mathbb{R}.v$ and $|v| = |\xi_t| = 1$. Let a be the point of intersection of $T_{x^+}\partial\Omega$ and $T_{x^-}\partial\Omega$. The vector $T^t v(w)$ always points to a, that is, $T^t v(w) \in \mathbb{R}.x_t a$. Thus,

$$F(T^t v(w)) = \frac{|T^t v(w)|}{2} \left(\frac{1}{|x_t y_t^+|} + \frac{1}{|x_t y_t^-|} \right),$$

where y_t^+ and y_t^- are the intersection points of (ax_t) and $\partial\Omega$. If $f : U \subset T_{x^+}\partial\Omega \longrightarrow \mathbb{R}$ denotes the function whose graph is a neighbourhood of x^+ in $\partial\Omega$, then

$$\frac{1}{2} \left(\frac{1}{|x_t y_t^+|} + \frac{1}{|x_t y_t^-|} \right) = \frac{1}{H(f_a^+, f_a^-)(|x_t x^+|)},$$

where f_a^+ and f_a^- are defined as in corollary 3.4.7. This corollary tells us that

$$\begin{aligned}
\limsup_{t \to +\infty} \frac{1}{t} \log \frac{1}{H(f_a^+, f_a^-)(|x_t x^+|)} &= \limsup_{t \to +\infty} - \frac{\log |x_t x^+|}{t} \frac{\log H(f_a^+, f_a^-)(|x_t x^+|)}{\log |x_t x^+|} \\
&= \limsup_{t \to +\infty} - \frac{\log |x_t x^+|}{t} \limsup_{s \to 0} \frac{\log H(f_a^+, f_a^-)(s)}{\log s} \\
&= \frac{2}{\underline{\alpha}(x^+, v)}
\end{aligned}$$

(recall that $|x_t x^+| \asymp e^{-2t}$). Hence

$$\limsup_{t \to +\infty} \frac{1}{t} \log F(T^t v(w)) = \frac{2}{\underline{\alpha}(x^+, v)} + \limsup_{t \to +\infty} \frac{1}{t} \log |T^t v(w)|.$$

From our choice of Euclidean metric, we have $|T^t v(w)| \asymp \langle T^t v(w), v\rangle$. Lemma 2.1.6 gives

$$T^t v(w) = -L_Y m(\varphi^t w)\xi_t + (m(w)m(\varphi^t w))^{1/2} d\pi(J^{X^e}(Y)),$$

where $Y \in VH\Omega$ is such that $d\pi(J^X(Y)) = v(w)$; $d\pi(J^{X^e}(Y))$ is collinear to v and has constant Euclidean norm, which implies that

$$\langle T^t v(w), v\rangle = (m(w)m(\varphi^t w))^{1/2} \asymp e^{-t}.$$

Hence

$$\overline{\eta}_+(w, v(w)) = \limsup_{t\to+\infty} \frac{1}{t} F(T^t v(w)) = \frac{2}{\underline{\alpha}(x^+, v)} - 1.$$

Obviously, the same holds for lower and backward exponents.

□

The last proposition tells us that the notions of Lyapunov regularity and exponents are projectively invariant, that is, it makes sense for codimension 1 submanifolds of $\mathbb{RP}^n$. It also implies the following

Corollary 3.4.10. *Let $f \in \mathtt{Cvx}(n)$. Then the numbers $\overline{\alpha}(v)$, $v \in \mathbb{R}^n \smallsetminus \{0\}$, can take only a finite numbers of values $+\infty \geqslant \alpha_1 > \cdots > \alpha_p \geqslant 1$, $1 \leqslant p \leqslant n$. The same holds for $\underline{\alpha}$. Moreover, the following propositions are equivalent:*

- *f is Lyapunov-regular;*
- *there exist a decomposition $\mathbb{R}^n = \oplus_{i=1}^p G_i$ and numbers $+\infty \geqslant \alpha_1 > \cdots > \alpha_p \geqslant 1$ such that the restriction $f|_{G_i \cap B(0,1)}$ is Lyapunov-regular with exponent α_i;*
- *there exist a filtration*
$$\{0\} = H_0 \subsetneq H_1 \subsetneq \cdots \subsetneq H_p = \mathbb{R}^n$$
and numbers $+\infty \geqslant \alpha_1 > \cdots > \alpha_p \geqslant 1$ such that, for any $v_i \in H_i \smallsetminus H_{i-1}$, the restriction $f|_{\mathbb{R}.v_i \cap B(0,1)}$ is Lyapunov-regular with exponent α_i.

When f is Lyapunov-regular, we call the numbers α_i the Lyapunov exponents of f.

Proof. The graph of f can always be considered as the boundary of a strictly convex set $\Omega \subset \mathbb{R}^{n+1}$ with C^1 boundary. We can then apply the last proposition to this Ω. □

Finally, we can state the definition of Lyapunov regularity for submanifolds of $\mathbb{RP}^n$:

Definition 3.4.11. *A locally strictly convex C^1 submanifold N of $\mathbb{RP}^n$ is said to be* **Lyapunov-regular** *at $x \in N$ if its trace in some (or, equivalently, any) affine chart at x is locally the graph of a Lyapunov regular function. The numbers $\alpha_1(x) \geqslant \cdots \geqslant \alpha_p(x)$ attached to x are called the* **Lyapunov exponents** *of x.*

The next proposition summarizes the results that will be useful later.

Proposition 3.4.12. *Let $w = (x, [\xi]) \in H\Omega$ be a forward regular point ending at x^+, with parallel Lyapunov exponents $-1 \leqslant \eta_1 < \cdots \leqslant \eta_p < 1$. Then $x^+ \in \partial\Omega$ is Lyapunov-regular with exponents*

$$\alpha_i = \frac{2}{\eta_i + 1},\ i = 1 \cdots p.$$

The Lyapunov decomposition of $T_{x^+}\partial\Omega$ is the projection of the Lyapunov decomposition of $T_x\mathcal{H}_w$ along xx^+.

3.5 Lyapunov manifolds

Proposition 3.4.12 allows us to define Lyapunov manifolds, that is, submanifolds tangent to the subspaces appearing in the Oseledets' filtration. In the classical theory of nonuniformly hyperbolic systems, the local existence of these manifolds is achieved with the help of Hadamard-Perron theorem (see [2]).

Choose an affine chart and a Euclidean metric on it such that Ω appears as a bounded subset of $\mathbb{R}^n$. Pick a Lyapunov regular point $x^+ \in \partial\Omega$ with at least one Lyapunov exponent > 1. Consider the (maybe noncomplete) Lyapunov filtration

$$\{0\} = H_0 \subsetneq H_1 \subsetneq \cdots \subsetneq H_p \subset T_{x^+}\partial\Omega,$$

corresponding to the Lyapunov exponents $\infty \geqslant \alpha_1 > \cdots > \alpha_p > 1$ which are strictly bigger than 1 (see corollary 3.4.10). This filtration is complete, that is, $H_p = T_{x^+}\partial\Omega$, if and only if 1 is not a Lyapunov exponent.
It induces the Lyapunov filtration

$$\{0\} = F_0(w) \subsetneq F_1(w) \subsetneq \cdots \subsetneq F_p(w) \subset T_x\mathcal{H}_w,$$

of $T_x\mathcal{H}_w$, for any $w = (x, [\xi])$ in the weak stable manifold

$$W^{cs}(x^+) = \{w = (x, [xx^+]),\ x \in \Omega\}$$

corresponding to x^+: if $v_i \in F_i(w) \smallsetminus F_{i-1}(w)$, we have

$$\lim_{t\to+\infty} \frac{1}{t} \log T^t_w v_i = \eta_i = \frac{2}{\alpha_i} - 1 < 1.$$

This filtration $(F_i(w))$ is nothing else than the projection on the basis Ω of the (noncomplete) Lyapunov filtration

$$\{0\} = F^s_0(w) \subsetneq F^s_1(w) \subsetneq \cdots \subsetneq F^s_p(w) \subset E^s(w)$$

of the stable subspace $E^s(w)$; here we have $F^s_i(w) = \oplus_{k=1}^i E^s_i(w)$, and $F^s_p(w)$ denotes the subspace of $E^s(w)$ consisting of vectors whose Lyapunov exponents are strictly negative (see section 3.2.1). In particular, any point $w \in W^{cs}(x^+)$ has the same negative forward Lyapunov exponents, which are given by

$$\chi_i^- = -1 + \eta_i = \frac{2}{\alpha_i} - 2.$$

Pick such a $w_0 = (x_0, [\xi_0]) \in W^{cs}(x^+)$. The horosphere $\mathcal{H}_{w_0}$ also admits a (noncomplete) filtration

$$\{x_0\} \subsetneq \mathcal{H}^1_{w_0} \subsetneq \cdots \subsetneq \mathcal{H}^p_{w_0} \subset \mathcal{H}_{w_0},$$

into C^1 submanifolds tangent to the $F_i(w)$, for $w \in W^s(w_0)$. These submanifolds are just defined by

$$\mathcal{H}^i_{w_0} = \mathcal{H}_{w_0} \cap (\mathbb{R}.\xi_0 \oplus H_i),$$

and it is easy to see that

$$\mathcal{H}^i_{w_0} = \{x \in \Omega,\ \limsup_{t\to+\infty} \frac{1}{t} \log d_\Omega(\pi\varphi^t(w_0), \pi\varphi^t(x, [xx^+])) \leqslant \chi_i^-\}.$$

They are the projections on Ω of the stable Lyapunov manifolds

$$W_i^s(w_0) := \{w = (x, [xx^+]),\ x \in \mathcal{H}^i_{w_0}\} = \{w \in H\Omega,\ \limsup_{t\to+\infty} \frac{1}{t} \log \overline{d}(\varphi^t(w_0), \varphi^t(w)) \leqslant \chi_i^-\},$$

which are tangent to the corresponding subspaces of the Lyapunov filtration of the stable distribution. In particular,

$$W_p^s(w_0) = \{w \in H\Omega,\ \limsup_{t\to+\infty} \frac{1}{t} \log \overline{d}(\varphi^t(w_0), \varphi^t(w)) < 0\}$$

Obviously, the same can be done for unstable distributions and manifolds: we get C^1 submanifolds

$$\{w_0\} \subsetneq W_p^u(w_0) \subsetneq \cdots \subsetneq W_1^u(w_0) \subset W^u(w_0),$$

of $W^u(w_0)$, where

$$W_i^u(w_0) = \{w \in H\Omega,\ \limsup_{t\to-\infty} \frac{1}{t} \log \overline{d}(\varphi^t(w_0), \varphi^t(w)) \geqslant \chi_i^+\}.$$

So, in particular,

$$W_1^u(w_0) = \{w \in H\Omega,\ \limsup_{t\to-\infty} \frac{1}{t} \log \overline{d}(\varphi^t(w_0), \varphi^t(w)) > 0\}.$$

3.6 Lyapunov exponents of a periodic orbit

We now consider a quotient manifold $M = \Omega/\Gamma$ and are interested in the Lyapunov exponents of a periodic orbit on HM. Every periodic orbit corresponds to a conjugacy class $[\gamma]$ of a hyperbolic element γ in the group Γ. Every such element is biproximal, that is: if $(\lambda_i)_{1\leqslant i\leqslant n}$ are its (non-necessary distinct) eigenvalues ordered as $|\lambda_1| \geqslant |\lambda_2| \cdots \geqslant |\lambda_{n+1}|$, then $|\lambda_1| > |\lambda_2|$ and $|\lambda_{n+1}| < |\lambda_n|$. The attractive fixed point of γ on $\partial\Omega$ is an eigenvector for the eigenvalue λ_1, and the repulsive one is an eigenvector for the eigenvalue λ_n. The length of the corresponding periodic orbit on M is given by

$$l(\gamma) = \frac{1}{2}(\log|\lambda_1| - \log|\lambda_{n+1}|).$$

Let us do the study in dimension 2. Take an element $\gamma \in \Gamma$ conjugated to the matrix

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \in SL_3(\mathbb{R}),$$

with $\lambda_i \in \mathbb{R},\ |\lambda_1| > |\lambda_2| > |\lambda_3|$. The line $(\gamma^-\gamma^+)$ is its axis and γ^0 its third fixed point. We look at the picture in the chart given by the plane $\{x_1 + x_3 = 0\} \subset \mathbb{R}^3$, with the following coordinates:

$$\gamma^- = [0 : 0 : 1],\ \gamma^+ = [1 : 0 : 0],\ \gamma^0 = [0 : 1 : 0].$$

This is a good chart for the periodic orbit from γ^- to γ^+ we are looking at. Choose a point $x \in (\gamma^-\gamma^+)$ with coordinates $[a_0 : 0 : 1 - a_0]$ where $a_0 \in (0,1)$ and let $w = (x, [\mathbf{x}\gamma^+])$. The point $x_n = \gamma^n.x$ is given by

$$x_n = [a_n : 0 : 1 - a_n],$$

with

$$a_{n+1} = \frac{\lambda_1 a_n}{\lambda_1 a_n + \lambda_2(1 - a_n)}.$$

Now, we look at a vector $v = \mathbf{xm} \in \gamma^-\gamma^{+\perp}$ with $m = [a_0 : b_0 : 1 - a_0]$, $b_0 \in \mathbb{R}$. Let $m_n = \gamma^n.m = [a_n : b_n : 1 - a_n]$, $v_n = \mathbf{x_n m_n}$, so that $|v_n| = |b_n|$. Then (b_n) is given by

$$b_{n+1} = \frac{\lambda_2 b_n}{\lambda_1 a_n + \lambda_2(1 - a_n)} = \frac{\lambda_2}{\lambda_1}\frac{a_{n+1}}{a_n} b_n,$$

which leads to

$$b_n = \left(\frac{\lambda_2}{\lambda_1}\right)^n \frac{b_0}{a_0} a_n.$$

Since $\lim_{n\to\infty} a_n = 1$, we get

$$b_n \asymp \left(\frac{\lambda_2}{\lambda_1}\right)^n.$$

Since γ is an isometry for F, we have, with the notations of lemma 2.4.4,

$$\begin{aligned} 1 \asymp F(x,v) = F(x_n, v_n) &\asymp \left|\frac{\lambda_2}{\lambda_1}\right|^n \frac{1}{|x_n\gamma^+|^{1/2}} \left(\frac{|x_n\gamma^+|^{1/2}}{|x_n y_n^+|} + \frac{|x_n\gamma^+|^{1/2}}{|x_n y_n^-|}\right) \\ &\asymp \left|\frac{\lambda_2}{\lambda_1}\right|^n e^{nl(\gamma)} F(T^{nl(\gamma)}(v)), \end{aligned}$$

by using lemma 2.4.3. Thus

$$F(T^{nl(\gamma)}(v)) \asymp \left|\frac{\lambda_1}{\lambda_2}\right|^n e^{-nl(\gamma)}$$

and

$$\lim_{t\to+\infty} \frac{1}{t}\log F(T^t(v)) = \lim_{n\to\infty} \frac{1}{nl(\gamma)} \log F(T^{nl(\gamma)}(v)) = -1 + 2\,\frac{\log|\lambda_1/\lambda_2|}{\log|\lambda_1/\lambda_3|}.$$

All this can be generalized to any dimension by sectioning the convex set, so that we get the following result.

Proposition 3.6.1. *Let γ be a periodic orbit of the flow, corresponding to a hyperbolic element $\gamma \in \Gamma$. Denote by $\lambda_0 > \lambda_1 > \cdots > \lambda_p > \lambda_{p+1}$ the moduli of the eigenvalues of γ. Then*

- *γ is regular and has no zero Lyapunov exponent;*
- *the Lyapunov exponents $(\eta_i(\gamma))$ of the parallel transport along γ are given by*

$$\eta_i(\gamma) = -1 + 2\,\frac{\log\lambda_0 - \log\lambda_i}{\log\lambda_0 - \log\lambda_{p+1}},\ i = 1\cdots p;$$

- *the sum of the parallel Lyapunov exponents is given by*

$$\eta(\gamma) = (n+1)\frac{\log \lambda_0 + \log \lambda_{p+1}}{\log \lambda_0 - \log \lambda_{p+1}}.$$

Proof. Only the last point remains to be proved. For that, we change the notation of the eigenvalues into $\lambda_0 \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_n$, where they are now counted with multiplicity. Then

$$\eta(\gamma) = \sum_{i=1}^{n-1} -1+2\,\frac{\log \lambda_0 - \log \lambda_i}{\log \lambda_0 - \log \lambda_n} = \sum_{i=1}^{n-1} \frac{\log \lambda_0 + \log \lambda_n - 2\log \lambda_i}{\log \lambda_0 - \log \lambda_n} = (n-1)\frac{\log \lambda_0 + \log \lambda_n}{\log \lambda_0 - \log \lambda_n} - 2\frac{\log \left(\prod_{i=1}^{n-1} \lambda_i\right)}{\log \lambda_0 - \log \lambda_n}.$$

Since $\gamma \in SL(n+1,\mathbb{R})$, that gives

$$\eta(\gamma) = (n+1)\frac{\log \lambda_0 + \log \lambda_n}{\log \lambda_0 - \log \lambda_n}.$$

$\square$

Chapter 4

Invariant measures

The preceding parts were approaching the geodesic flow of Hilbert metrics from a topological or differential point of view. We now turn to the measure or ergodic point of view, that is, we look at our dynamical system endowed with an invariant probability measure. We are especially interested in the classical theory of Patterson-Sullivan measures and we extend here various results from hyperbolic geometry.

4.1 Generalities

Ergodic theory looks at dynamical systems from a measure point of view. It considers the measurable action of a group G on a measure space $(X, \mathcal{A}, \mu)$, which preserves the Radon measure μ: for any $g \in G$, $g * \mu = \mu$, that is, for any $A \in \mathcal{A}$, $\mu(g^{-1}A) = \mu(A)$. The measure is often assumed to have total mass 1; this assumption can be seen as a measurable counterpart of the compactness of the space, which is often assumed when studying dynamical systems from a topological point of view.

In this chapter, we use this approach to study our geodesic flow. It is not defined on a compact space, but we can still hope to find invariant probability measures, which would turn the space into a finite one, from this new point of view. Of course, any invariant measure does not give an interesting information on the system. For example, the uniform Lebesgue measure carried by a periodic orbit is not in itself very interesting, for it sees only what occurs on the periodic orbit, where the dynamic is trivial.

4.1.1 The Kaimanovich correspondence

Let $M = \Omega/\Gamma$ be the quotient manifold of a strictly convex proper open set Ω with C^1 boundary by a nonelementary group $\Gamma \subset Isom(\Omega, d_\Omega)$. Consider the geodesic flow φ^t of the Hilbert metric on HM. φ^t is continuous and thus Borel-measurable, the Borel σ-algebra $\mathcal{B}$ being the one generated by open subsets of HM.
Let $\mathcal{M}$ denote the set of Borel φ^t-invariant probability measures on HM. $\mathcal{M}$ is a convex set, and is nonempty: since Γ is nonelementary, it contains a hyperbolic element, hence there exist periodic orbits, and $\mathcal{M}$ contains all the Lebesgue measures carried by these periodic orbits; $\mathcal{M}$ even contains the convex hull of such measures.

We endow $\mathcal{M}$ with the topology of weak convergence of measures: a sequence (μ_n) of measures converges to μ if, for any continuous function $f : HM \longrightarrow \mathbb{R}$,

$$\lim_{n\to+\infty} \int f \ d\mu_n = \int f \ d\mu.$$

For this topology, $\mathcal{M}$ is compact.
The extremal set of $\mathcal{M}$ consists in **ergodic** measures. Ergodic measures are those measures for which any invariant Borel set has either full or zero measure. The measures carried by periodic orbits are ergodic, hence lie on the extremal set of $\mathcal{M}$. Under certain hypotheses, the set of measures carried by periodic orbits is dense inside the set of ergodic measures. A theorem of Coudène and Schapira [24] says it suffices to prove an Anosov closing lemma, which is easy to prove in our context.

The interest in ergodic measures lies in the following theorem, known as Birkhoff ergodic theorem:

Theorem 4.1.1. *Let μ be an invariant probability measure for the flow φ^t on X. Then, for any function $f \in L^1(X,\mu)$, the limit*

$$F(x) := \lim_{T\to+\infty} \frac{1}{T} \int_0^T f(\varphi^t(x)) \ dt$$

exists for μ-almost every point $x \in X$ and moreover, $\int F \ d\mu = \int f \ d\mu$. In particular, if μ is ergodic then

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T f(\varphi^t(x)) \ dt = \int f \ d\mu \tag{4.1}$$

μ-almost everywhere.

This result means that if μ is ergodic, then the space averages with respect to μ describe the asymptotical time averages. Note the following important fact: let μ be an ergodic measure for a flow φ^t on a space X; if f is a φ^t-invariant measurable function on X, then it is constant μ-almost everywhere.

The first thing we will see is that there is a natural correspondence between the dynamics of φ^t on HM and the dynamics of the action of Γ by coordinates on the double boundary $\partial^2\Omega = \partial\Omega\times\partial\Omega\smallsetminus\Delta$, where $\Delta = \{(x,x), \ x \in \partial\Omega\}$ denotes the diagonal. This correspondence is easy and relies on the fact that $\partial^2\Omega$ is nothing else than the space of oriented geodesics of Ω: to each oriented geodesic $\gamma : \mathbb{R} \longrightarrow \Omega$, we can associate the pair (x^+, x^-) consisting ot its two endpoints $x^+ = \gamma(+\infty)$ and $x^- = \gamma(-\infty)$; then, the action of φ^t on a geodesic $\gamma : \mathbb{R} \longrightarrow \Omega$ is just a translation, and, when we forget about it, we get the double boundary $\partial^2\Omega$. Clearly, this construction does not work anymore when the convex set is not strictly convex.

The main results are the following theorem and its corollary, which establish the expected correspondence about invariant Radon measures under the action of φ^t on HM and of the group Γ on $\partial^2\Omega$ by coordinates. It was proved by Kaimanovich in [44] and his proof clearly works in the present case. Basically, it relies on the observation we just made.

Theorem 4.1.2 (Kaimanovich [44])**.** *Let Ω be a strictly convex proper open set with C^1 boundary. There is a convex isomorphism between the cone of Radon measures on $\partial^2\Omega$ and the cone of Radon measures on $H\Omega$ invariant under the geodesic flow.*

Proof. Let us just recall, without justification, how we pass from a measure Λ on $\partial^2\Omega$ to λ on $H\Omega$ and conversely:

- If Λ is given, we define λ by setting, for any Borel subset $A \subset H\Omega$,
$$\lambda(A) = \int_{\partial^2\Omega} l((\xi^-\xi^+) \cap A)\ d\Lambda(\xi^-, \xi^+),$$
where $l((\xi^-\xi^+) \cap A)$ denotes the Hilbert length of the intersection of the line $(\xi^-\xi^+)$ with A.
- If λ is given and K is a compact Borel subset of $\partial^2\Omega$, we decompose its preimage $p^{-1}(K) \subset H\Omega$ by $p : w \in H\Omega \longmapsto (x^+, x^-) = (\varphi^{+\infty}(w), \varphi^{-\infty}(w))$, as a union $\cup_{n\in\mathbb{Z}}\varphi^n(K_0)$ of $(mod\ 0)$ disjoint compact subsets "of length 1", and set $\Lambda(K) = \lambda(K_0)$.

□

Corollary 4.1.3 (Kaimanovich [44])**.** *Let $M = \Omega/\Gamma$ be the quotient manifold of a strictly convex proper open set Ω with C^1 boundary by a nonelementary group $\Gamma \subset Isom(\Omega, d_\Omega)$. Then there is a convex isomorphism between the cone of Γ-invariant Radon measures on $\partial^2\Omega$ and the cone of Radon measures on HM invariant under the geodesic flow. This isomorphism preserves ergodicity.*

The flip map at infinity is the involution $\partial\sigma$ of $\partial^2\Omega$ defined by $\partial\sigma(\xi, \eta) = (\eta, \xi)$. It is a straightforward observation that the correspondence of theorem 4.1.2 is flip invariant: if λ on $H\Omega$ corresponds to Λ on $\partial^2\Omega$, then $\sigma * \lambda$ corresponds to $\partial\sigma * \Lambda$.

4.1.2 Measure-theoretic entropy

The topological entropy is a measure of the topological complexity of a transformation $\Phi : X \longrightarrow X$ of a metric space (X, d). The measure-theoretic entropy plays the same role for a tranformation $\Phi : X \longrightarrow X$ of a probability space $(X, \mathcal{A}, \mu)$. By a transformation (or a morphism), we mean a measurable map which preserves the measure μ. Measure-theoretic entropy was defined before topological entropy by Kolmogorov and then revisited by Sinai. We refer to the classical books [74], [62] or [47] for more details.

A countable partition P of a probability space $(X, \mathcal{A}, \mu)$ is a collection $(P_i)_{i\in\mathbb{N}}$ of measurable subsets of X such that
$$\mu(P_i \cap P_j) = 0, \quad \mu(X \smallsetminus \cup_{i\in\mathbb{N}} P_i) = 0.$$
An element P_i of P is called an atom of P. To almost any $x \in X$ can be associated the atom $P(x)$ of P containing x; the function $x \longmapsto P(x)$ is measurable.

The entropy of such a partition is defined as
$$H(P) = -\sum_{i\in\mathbb{N}} \mu(P_i) \log\ \mu(P_i). \tag{4.2}$$

It represents the information given by the partition P on (X, μ): it gives a measure of how precise in average is the information that a point x is in the atom P_i of P. For example, if P is the partition in one atom consisting of X, then $H(P) = 0$: we do not know more on the position of a point $x \in X$

if we know that x is in X...

Now consider a transformation $\Phi : (X, \mu) \longmapsto (X, \mu)$. Given a partition P, we want to see how Φ transforms this partition; this is measured by the average entropy of P under T. Φ transforms the partition P in a new partition ΦP whose atoms are the $\Phi^{-1}(P_i)$. Let P^n be the joint partition

$$P^n = \bigvee_{i=0}^{n-1} \Phi^i P;$$

$P \vee Q$ denotes the joint partition

$$P \vee Q = \{A \cap B,\ A \in P, B \in Q\}.$$

The atom $\Phi P(x)$ containing x is $\Phi^{-1}(P(\Phi x))$. The atom $P^n(x)$ containing x is the intersection

$$P^n(x) = P(x) \cap \Phi^{-1}(P(\Phi x)) \cap \cdots \cap \Phi^{n-1}(P(\Phi^{n-1} x)).$$

For example, if Φ is an Anosov diffeomorphism, this intersection tends to consist of little pieces of stable manifolds. This remark will be crucial in the next chapter.

The average entropy $h(P, \Phi)$ of P under the T is defined by

$$h(P, \Phi) = \lim_{n\to\infty} \frac{1}{n} H(P^n). \tag{4.3}$$

The measure-theoretic entropy of Φ is then the supremum

$$h(\Phi) = \sup_P h(P, \Phi),$$

which is taken with respect to all finite, or countable partitions with finite entropy. A partition which would achieve this supremum is in some sense well adapted to describe the action of Φ. Kolmogorov and later Sinai showed that generating partitions are such adapted partitions. By a **generating partition**, we mean a partition P such that

$$\bigvee_{n=-\infty}^{+\infty} \Phi^n P$$

is the partition by points. However, the existence of generating partition was not clear until Rokhlin proved

Theorem 4.1.4 (Rokhlin, [69])**.** *Let Φ be a transformation of a probability space (X, μ), with finite entropy. If Φ is aperiodic, that is, the measure of periodic points is 0, then Φ admits a countable generating partition of finite entropy.*

For a flow $\varphi = (\varphi^t)$ on some probability space (X, μ) which preserves μ, the measure-theoretic entropy is defined as the entropy of the time-one map: $h(\varphi) := h(\varphi^1)$. The identity $h(\varphi^s) = |s| h(\varphi^1)$ for $s \neq 0$ justifies this definition.

A general Borel map $\Phi : X \longrightarrow X$ have lots of invariant probability measures, and we can consider the entropy of each of these measures. In this case, we index all the entropies by the measure μ: $h_\mu(\Phi)$, $h_\mu(P,\Phi)$... The essential result is the following theorem, known as variational principle, which asserts that topological entropy is the supremum of measure-theoretic entropies. It was first proved by Goodman [36] for the classical definition on compact spaces; Misiurewicz [58] then gave a simplified proof. The generalization to more general spaces is due to Handel and Kitchens [39] and uses the result in the compact case.

Theorem 4.1.5 (Variational Principle)**.** *Let $\Phi : X \longrightarrow X$ be a homeomorphism of a locally compact metric space X and $\mathcal{M}$ be the set of Φ-invariant probability measures. Then*

$$h_{top}(\Phi) = \sup_{\mu\in\mathcal{M}} h_\mu(\Phi).$$

A measure which achieves the supremum in the variational principle is called a *measure of maximal entropy*.

4.2 Conformal densities and Bowen-Margulis measures

We get now interested in the most popular invariant measures on negatively curved manifolds: the family (μ_x) of Patterson-Sullivan measures on the boundary at infinity, whose double $\mu_x \otimes \mu_x$, renormalized by a factor to make it Γ invariant, is associated to the Bowen-Margulis measure on HM. Nothing new appears in our context, so we mainly recall the already known results and constructions made for pinched negatively curved manifolds or CAT(-1) spaces.

4.2.1 Conformal densities

A conformal density of dimension δ is a family of measures $(\nu_x)_{x\in\Omega}$ on $\partial\Omega$ all in the same class, and such that

$$\frac{d\nu_x}{d\nu_y}(\xi) = e^{-\delta b_\xi(x,y)}.$$

The family $(\nu_x)_{x\in\Omega}$ is said to be Γ-invariant if $\nu_{gx} = g * \nu_x$.
The Poincaré series of Γ is the series defined by

$$g_\Gamma(s,x) = \sum_{\gamma\in\Gamma} e^{-s d_\Omega(x,\gamma o)},$$

where o denotes some fixed base point. δ_Γ denotes the critical exponent of this series: for $s < \delta_\Gamma$, the series diverges, and for $s > \delta_\Gamma$, it converges; at $s = \delta_\Gamma$, both are possible and we will see that this plays a crucial role in the theory. We say that Γ is **divergent** if the Poincaré series diverges at the critical exponent, and **convergent** otherwise.
For $t > 0$, let

$$N_\Gamma(o,R) = \sharp\{\gamma,\ d_\Omega(o,\gamma o) < R\}.$$

Then we have

$$\delta_\Gamma = \limsup_{R\to+\infty} \frac{1}{R} \log N_\Gamma(o,R).$$

Theorem 4.2.1 (Patterson, Sullivan)**.** *Let Γ be a nonelementary discrete subgroup of $Isom(\Omega, d_\Omega)$ and δ_Γ be its critical exponent. Then there exists a conformal density $(\mu_x)_{x\in\Omega}$ of dimension δ_Γ.*

Proof. We make a sketch of the proof given by Patterson for convenience, and also because we will need some technical details later. Fix $o \in \Omega$. Consider the measures μ_x^s for $x \in \Omega$ and $s > \delta_\Gamma$, defined by

$$\mu_x^s = \frac{1}{g_\Gamma(s,o)} \sum_{\gamma\in\Gamma} e^{-sd_\Omega(x,\gamma o)} \delta_{\gamma o}.$$

These are finite measures supported on $\Gamma.o$; the family $(\mu_x^s)_x$ is Γ-invariant: for any Borel subset $A \subset \Omega$ and any $g \in \Gamma$,

$$\mu_x^s(g^{-1}A) = \frac{1}{g_\Gamma(s,o)} \sum_{\gamma\in\Gamma} e^{-sd_\Omega(x,\gamma o)} \delta_{\gamma o}(g^{-1}A) = \frac{1}{g_\Gamma(s,o)} \sum_{\gamma\in\Gamma} e^{-sd_\Omega(gx,g\gamma o)} \delta_{g\gamma o}(A) = \mu_{gx}^s(A);$$

and for two different points x and y, we have

$$\frac{d\mu_x^s}{d\mu_y^s}(\gamma o) = e^{-s(d_\Omega(x,\gamma o) - d_\Omega(y,\gamma o))} := e^{-sb_{\gamma o}(x,y)}.$$

If we consider these measures $\mu_x^s,\ x \in \Omega, s > \delta_\Gamma$ as measures on $\overline{\Omega}$, then we can write, for any $z \in \overline{\Omega}$,

$$\frac{d\mu_x^s}{d\mu_y^s}(z) = e^{-sb_z(x,y)}; \tag{4.4}$$

The function $z \in \overline{\Omega} \mapsto b_z(x,y)$ is continuous on $\overline{\Omega}$ and coincide with the Busemann function when $z \in \partial\Omega$. For some $x \in \Omega$, let μ_x be a weak limit of μ_x^s when s decreases to δ_Γ, following some subsequence $(s_n)_{n\in\mathbb{N}}$. Equation (4.4) implies that the corresponding limits $\mu_y = \lim_{n\to\infty} \mu_y^{s_n}$ are well defined. All these measures are supported on $\overline{\Gamma.o}$, the family $(\mu_x)_{s\in\Omega}$ is Γ-invariant, and for $\xi \in \partial\Omega$,

$$\frac{d\mu_x}{d\mu_y}(\xi) = e^{-\delta b_\xi(x,y)}.$$

So we are almost done. In fact, we are done if we assume that the Poincaré series diverges at δ_Γ: in this case, the measures are supported on $\Lambda_\Gamma = \overline{\Gamma.o} \smallsetminus \Gamma.o$. When the Poincaré series converges at δ_Γ, Patterson explained that we can make it diverge using an auxiliary function that does not change the critical exponent. That is, we replace the Poincaré series by

$$g'_\Gamma(s,x) = \sum_{\gamma\in\Gamma} h(d_\Omega(x,\gamma o)) e^{-sd(x,\gamma o)},$$

where $h : \mathbb{R}_+ \to \mathbb{R}_+$ is some increasing function whose growth is subexponential, that is, for any $\eta > 0$, there exists $t_\eta > 0$, such that for $t > t_\eta$,

$$h(t+s) \leqslant h(t)e^{\eta s}.$$

$\square$

From now on, a Γ-invariant conformal density $(\mu_x)_{x\in\Omega}$ of dimension δ_Γ will also be called a **Patterson-Sullivan density**, and one individual measure μ_x a **Patterson-Sullivan measure**.

Lemma 4.2.2 (Sullivan's shadow lemma)**.** *Let (μ_x) be a conformal Γ-invariant density of dimension δ. For any $x \in \Omega$ and r large enough, there exists $C_{x,r} > 0$ such that for any $\gamma \in \Gamma$*

$$\frac{1}{C_{x,r}} e^{-\delta d_\Omega(x,\gamma x)} \leqslant \mu_x(\mathcal{O}_r(x,\gamma x)) \leqslant C_{x,r} e^{-\delta d_\Omega(x,\gamma x)}$$

Proof. Here comes Roblin's proof in [67]. We have

$$\mu_x(\mathcal{O}_r(x,\gamma x)) = \mu_x(\gamma \mathcal{O}_r(\gamma^{-1}x, x)) = \mu_{\gamma^{-1}x}(\mathcal{O}_r(\gamma^{-1}x, x)) = \int_{\mathcal{O}_r(\gamma^{-1}x,x)} e^{-\delta b_\xi(\gamma^{-1}x,x)}\ d\mu_x(\xi).$$

From lemma 1.2.1, we have that

$$e^{-\delta d_\Omega(\gamma^{-1}x,x)} \leqslant e^{-\delta b_\xi(\gamma^{-1}x,x)} \leqslant e^{-\delta(d_\Omega(\gamma^{-1}x,x)-2r)},$$

hence

$$e^{-\delta d_\Omega(x,\gamma x)} \mu_x(\mathcal{O}_r(\gamma^{-1}x,x)) \leqslant \mu_x(\mathcal{O}_r(x,\gamma x)) \leqslant e^{2\delta r} e^{-\delta d_\Omega(x,\gamma x)} \mu_x(\mathcal{O}_r(\gamma^{-1}x,x)).$$

Now, just remark that $\mu_x(\mathcal{O}_r(\gamma^{-1}x,x)) \leqslant \mu_x(\partial\Omega)$ to get the result. $\square$

This lemma admits the following

Corollary 4.2.3 (Sullivan)**.** *Let Γ be a nonelementary discrete subgroup of $Isom(\Omega, d_\Omega)$ and δ_Γ be its critical exponent.*

- *If there exists a conformal Γ-invariant density of dimension δ, then $\delta \geqslant \delta_\Gamma$.*
- *For each $o \in \Omega$, there exists some $C_o > 0$ such that*

$$N_\Gamma(o,R) \leqslant C_o e^{\delta_\Gamma R}.$$

4.2.2 Bowen-Margulis measures

The Bowen-Margulis measure of a topologically mixing Anosov flow (or diffeomorphism) is the unique measure of maximal entropy, that is, the unique measure which achieves the supremum in the variational principle of theorem 4.1.5. It was first constructed by Margulis in his PhD thesis for the geodesic flow of negatively curved manifolds (c.f. [52, 53]). In [15, 16], Bowen proved that, on a closed hyperbolic manifold, closed geodesics were uniformly distributed with respect to the Liouville measure. Bowen's construction extends to the case of a topologically mixing Anosov flow, and finally, one finds that closed orbits are uniformly distributed with respect to a specific measure, which indeed coincides with the measure constructed by Margulis. So the name of the measure.
A striking consequence of Margulis's construction is the precise asymptotic expansion of the number $N(t)$ of primitive closed orbits of length at most t, which was given by Margulis (see [53] or [47]):

$$N(t) \sim \frac{e^{-ht}}{ht},$$

where h denotes the topological entropy of the topologically mixing Anosov flow under consideration.

A general construction

To each Γ-invariant conformal density on $\partial\Omega$, one can construct a φ^t-invariant measure on HM by a process that we now describe. It can be found in Sullivan [72]. When M is compact, this construction allows to recover the Bowen-Margulis measure from the Patterson-Sullivan measures.

Let (μ_x) be a conformal density of dimension δ. Consider the product measure $\mu_x^2 = \mu_x \otimes \mu_x$ on $\partial^2\Omega$. We have

$$\begin{aligned} d(g\mu_x^2)(\xi^+,\xi^-) = d\mu_{gx}^2(\xi^+,\xi^-) &= e^{-\delta(b_{\xi^+}(gx,x)+b_{\xi^-}(gx,x))} d\mu_x^2(\xi^+,\xi^-) \\ &= e^{-2\delta((\xi^+|\xi^-)_{gx}-(\xi^+|\xi^-)_x)} d\mu_x^2(\xi^+,\xi^-). \end{aligned}$$

Thus letting

$$d\Lambda_x(\xi^+,\xi^-) = e^{2\delta(\xi^+|\xi^-)_x} d\mu_x^2(\xi^+,\xi^-),$$

we get a Γ-invariant measure on $\partial^2\Omega$. In fact, this measure Λ_x does not depend on x:

$$d\Lambda_x(\xi^+,\xi^-) = e^{2\delta(\xi^+|\xi^-)_x} d\mu_x^2(\xi^+,\xi^-) = e^{2\delta(\xi^+|\xi^-)_x} e^{-\delta(b_{\xi^+}(x,y)+b_{\xi^-}(x,y))} d\mu_y^2(\xi^+,\xi^-)$$

and

$$\begin{aligned} 2(\xi^+|\xi^-)_x - b_{\xi^+}(x,y) - b_{\xi^-}(x,y) = &\ \lim_{z^\pm\to\xi^\pm} \begin{array}{r} d_\Omega(x,z^+) + d_\Omega(x,z^-) - d_\Omega(z^+,z^-) \\ -d_\Omega(x,z^+) + d(y,z^+) - d_\Omega(x,z^-) + d_\Omega(y,z^-) \end{array} \\ = &\ 2(\xi^+|\xi^-)_y, \end{aligned}$$

so that

$$d\Lambda_x(\xi^+,\xi^-) = d\Lambda_y(\xi^+,\xi^-)$$

Theorem 4.1.2 tells us that to Λ_x is associated an invariant measure μ of the geodesic flow on HM. This measure μ inherits strong properties:

- μ is flip invariant since by construction, Λ_x is flip-invariant;
- μ has a local product structure, that is μ is locally the product $\mu = \mu^s \otimes \mu^u \otimes dt$, where μ^u and μ^s denote the stable and unstable conditional measures of μ;
- μ^s and μ^u are naturally related to the measures μ_x. In fact, any stable or unstable leaf can be identified with some $\mathcal{H} \smallsetminus \{p\}$, where $\mathcal{H}$ is a horosphere based at p, and by projection, μ^s and μ^u can be seen as measures on $\partial\Omega \smallsetminus \{p\}$, which are in the same Lebesgue class as μ_x. From this, we get the important transition property of the conditional measures: for all $t \in \mathbb{R}$ and $w \in HM$,
$$\varphi^t * \mu_w^s = e^{-\delta t}\mu_{\varphi^t(w)}^s,\ \varphi^t * \mu_w^u = e^{\delta t}\mu_{\varphi^t(w)}^u.$$

Hopf-Tsuji-Sullivan theorem

The main result about conformal densities and the associated measures is the following theorem, known as Hopf-Tsuji-Sullivan theorem. It has a long history and I am certainly not aware of all the steps. The most achieved version, that we state here, is due to Roblin in the beautiful [67]: he proved it in the context of $CAT(-1)$ spaces, and his proof works without any change in our context. The main reason for this adaptation to be possible is that he never uses angle considerations; instead, he essentially works with shadows of balls at infinity. In [45], Kaimanovich had already given a part of the result for some non-Riemannian spaces, but also for more general families of measures. Sullivan was the first to be really involved in this kind of questions, but he was essentially working in the hyperbolic space, where it is possible to go deeper; in particular, Sullivan always made links with spectral theory, which is a priori not relevant in the case of non-Riemannian spaces.

Theorem 4.2.4 (Hopf, Tsuji, Sullivan, Kaimanovich, Roblin...). *Let (μ_x) be a Γ-invariant conformal density of dimension δ, Λ and μ the associated measures on $\partial^2\Omega$ and HM. Denote by Λ_r the set of radial limit points. Fix any $x \in \Omega$. Then either*

1. $\displaystyle\sum_{\gamma\in\Gamma} e^{-\delta d_\Omega(x,\gamma x)} = +\infty$;
2. *Λ_r has full μ_x-measure;*
3. *Λ is ergodic for the action of Γ on $\partial^2\Omega$;*
4. *μ is ergodic for the geodesic flow on HM;*

or

1. $\displaystyle\sum_{\gamma\in\Gamma} e^{-\delta d_\Omega(x,\gamma x)} < +\infty$;
2. $\mu_x(\Lambda_r) = 0$;
3. *Λ is completely dissipative for the action of Γ on $\partial^2\Omega$;*
4. *μ is completely dissipative for the geodesic flow on HM.*

To understand the theorem, we have to recall the definitions of a conservative and dissipative measures. Consider the μ-preserving action of a group G (Γ of $\mathbb{R}$ in the last theorem) on some measure space (X,μ). A wandering set A is a measurable set such that all its translates by G are disjoint *mod* 0, that is, for two distinct elements $g, g' \in G$, $\mu(gA \cap g'A) = 0$. The measure μ is then called **conservative** if every non-trivial measurable set A is nonwandering, and **completely dissipative** if it admits a wandering set A such that $\Gamma.A$ has full measure.

Poincaré recurrence theorem states that any finite measure is conservative. Unless the space consists of a unique dissipative orbit, ergodicity always implies conservativity but the converse is not true for general dynamical systems. A crucial part in the proof of theorem 4.2.4 is the following

Lemma 4.2.5. *The measure μ is conservative if and only if it is ergodic.*

Bowen-Margulis measures

A measure μ on HM associated to a Patterson-Sullivan density (μ_x) will be called a **Bowen-Margulis measure**. It is straightforward from the construction that two Patterson-Sullivan densities are in the same Lebesgue class if and only if the Bowen-Margulis are so.

Corollary 4.2.6. *If Γ is divergent, then all Bowen-Margulis measures are proportional.*

Proof. Let (ν_x) and (μ_x) be two Γ-invariant conformal densities of dimension δ_Γ. Since Γ is divergent, that is the Poincaré series diverges at δ_Γ, we are in the first alternative of theorem 4.2.4. The family $(\lambda_x) = (\frac{1}{2}(\nu_x + \mu_x))$ is also a Patterson-Sullivan measure, hence theorem 4.2.4 says that the action of Γ on $\partial^2\Omega$ is ergodic with respect to some measure in the class of $\lambda_x \otimes \lambda_x$. But this is a contradiction since $\lambda_x \otimes \lambda_x$ is the middle of $\mu_x \otimes \mu_x$ and $\nu_x \otimes \nu_x$; unless μ_x and ν_x are in the same class.
From the observation above, this implies that all Bowen-Margulis measures are in the same class. Since they are ergodic, they are indeed all proportional. $\square$

In the case M is compact, the group is always divergent and we recover in this way the measure of maximal entropy constructed by Bowen and Margulis. So the name... The conditional measures μ^s and μ^u along stable and unstable manifolds will be called the Margulis measures, because these were central in Margulis construction of the measure. Let us recall their essential transition property:

$$\forall w \in HM,\ \forall t \in \mathbb{R},\ \varphi^t * \mu^s_w = e^{-\delta_\Gamma t}\mu^s_{\varphi^t(w)},\ \varphi^t * \mu^u_w = e^{\delta_\Gamma t}\mu^u_{\varphi^t(w)}.$$

To check that Γ is divergent is often not an easy thing to do. The second point of theorem 4.2.4, about the mass of the radial limit set Λ_r, is easier to check as we will see in the next section. A special case is given by the following

Corollary 4.2.7. *Let Γ be a nonelementary group. If some Bowen-Margulis measure μ is finite, then Γ is divergent.*

Proof. If μ is finite, then it is conservative and we are thus in the first part of theorem 4.2.4. $\square$

Nevertheless, remark that there exist pinched negatively curved manifolds $M = \tilde{M}/\Gamma$ with Γ divergent, but whose Bowen-Margulis measures are all infinite. Some examples were given by Pollicott and Sharp [66], and geometrically finite ones have been recently constructed by Peigné [64].

4.3 Geometrically finite surfaces

The goal of this section is to prove the following

Theorem 4.3.1. *Let $M = \Omega/\Gamma$ be a geometrically finite surface. Then there is a finite Bowen-Margulis measure on HM.*

(All of this works in higher dimensions as well thanks to the results of [26].)
The proof of the theorem will take some time, and we will prove some intermediate results which are of interest. This development is very classic, and can be already found in [72]. The proofs provided here are largely inspired by an unpublished paper of M. Peigné [63], available on his webpage.

We begin by an obvious observation.

Lemma 4.3.2. *Let Γ act on Ω and Ω' with $\Omega \subset \Omega'$. Denote by $g_{\Gamma,\Omega}(s,x)$ and $g_{\Gamma,\Omega'}(s,x)$ the Poincaré series for the action of Γ on Ω and Ω', and $\delta_\Gamma(\Omega)$ and $\delta_\Gamma(\Omega')$ theire critical exponent. Then, for any $s > \delta_\Gamma(\Omega')$, $g_{\Gamma,\Omega}(s,x) \leqslant g_{\Gamma,\Omega'}(s,x)$. In particular, $\delta_\Gamma(\Omega) \leqslant \delta_\Gamma(\Omega')$.*

Proof. If $x, y \in \Omega$ then $d_{\Omega'}(x,y) \leqslant d_\Omega(x,y)$. So, if $x \in \Omega$ and $s > \delta_\Gamma(\Omega')$, we have $g_{\Gamma,\Omega}(s,x) \leqslant g_{\Gamma,\Omega'}(s,x)$. In particular, the convergence of $g_{\Gamma,\Omega'}(s,x)$ implies the convergence of $g_{\Gamma,\Omega}(s,x)$, hence the result. $\square$

Lemma 4.3.3. *Let $\Omega \subset \mathbb{RP}^2$. The critical exponent of a discrete parabolic subgroup $\mathcal{P}$ is $\delta_\mathcal{P} = \frac{1}{2}$ and the Poincaré series of $\mathcal{P}$ diverges at $\delta_\mathcal{P}$.*

Proof. Call p the fixed point of $\mathcal{P}$. As remarked in lemma 1.3.4, we can find two $\mathcal{P}$-invariant ellipses $\mathcal{E}^{int}$ and $\mathcal{E}^{ext}$ containing p in their boundary such that $\mathcal{E}^{int} \subset \Omega \subset \mathcal{E}^{ext}$. Now we know from hyperbolic geometry that $\delta_\mathcal{P}(\mathcal{E}^{int}) = \delta_\mathcal{P}(\mathcal{E}^{ext}) = \frac{1}{2}$ and that the Poincaré series diverges at the critical exponent. From lemma 4.3.2, the same holds for $\mathcal{P}$ acting on Ω. $\square$

Lemma 4.3.4. *If a nonelementary group Γ acting on Ω contains a parabolic subgroup, then $\delta_\Gamma > \frac{1}{2}$.*

Proof. From lemma 4.3.3, we get $\delta_\Gamma \geqslant \frac{1}{2}$, so we just have to prove that the inequality is strict. Let ξ be the fixed point of $\mathcal{P}$. Since Γ is nonelementary, we can find a hyperbolic element $h \in \Gamma$ such that Γ contains the group $H * \mathcal{P}$ where $H = \langle h \rangle$: this is a classical ping-pong argument. In particular, G contains all the distinct elements $g = h^{n_1}p_1 \cdots h^{n_l}p_l$ for $l \geqslant 1,\ n_i \geqslant 1,\ p_i \in \mathcal{P} \smallsetminus \{Id\}$. So,

$$\begin{aligned}
g_\Gamma(s,x) = \sum_{g\in\Gamma} e^{-sd_\Omega(x,gx)} &\geqslant \sum_{l\geqslant 1} \sum_{\substack{n_1,\cdots n_l,\\ p_1,\cdots,p_l}} e^{-sd_\Omega(x,h^{n_1}p_1\cdots h^{n_l}p_l x)} \\
&\geqslant \sum_{l\geqslant 1} \sum_{\substack{n_1,\cdots n_l,\\ p_1,\cdots,p_l}} e^{-sd_\Omega(x,h^{n_1}x)} e^{-sd_\Omega(x,p_1x)} \cdots e^{-sd_\Omega(x,h^{n_l}x)} e^{-sd_\Omega(x,p_lx)} \\
&= \sum_{l\geqslant 1} \left(\left(\sum_{n\in\mathbb{Z}} e^{-sd_\Omega(x,h^n x)} \right) \left(\sum_{p\in\mathcal{P}\smallsetminus\{Id\}} e^{-sd_\Omega(x,px)} \right) \right)^l \\
&= \sum_{l\geqslant 1} (g_H(s,x)(g_\mathcal{P}(s,x)-1))^l.
\end{aligned}$$

But $g_H(s,x)$ converges for any $s > 0$ and $g_\mathcal{P}(s,x)$ converges for $s > \frac{1}{2}$ and diverges for $s = \frac{1}{2}$. So there exists $s_0 > \frac{1}{2}$ for which $g_H(s,x)(g_\mathcal{P}(s,x)-1) > 1$, so that $g_\Gamma(s_0,x)$ diverges. Hence $\delta_\Gamma \geqslant s_0 > \frac{1}{2}$. $\square$

Proposition 4.3.5. *Let $M = \Omega/\Gamma$ be a geometrically finite surface. Then any Patterson-Sullivan measure has no atom.*

Proof. Let $o \in \Omega$ and let (μ_x) be a family of Patterson-Sullivan measures: μ_x is obtained as a weak limit of the family $(\mu_x^s)_{s>\delta_\Gamma}$ where

$$\mu_x^s = \frac{1}{g'_\Gamma(s,o)} \sum_{\gamma\in\Gamma} h(d(o,\gamma o)) e^{-sd(x,\gamma o)} \delta_{\gamma o},$$

and

$$g'_\Gamma(s,o) = \sum_{\gamma\in\Gamma} h(d_\Omega(o,\gamma o))e^{-sd_\Omega(o,\gamma o)}.$$

Since all these measures μ_x^s are in the same class, we just have to prove the result for $\mu_o = \lim \mu_o^s$, so we abbreviate by $\mu := \mu_o$ and $\mu^s = \mu_o^s$.

First of all, remark that Sullivan's shadow lemma 4.2.2 implies that μ has no atom on the radial limit set Λ_r. Since $\Lambda \smallsetminus \Lambda_r$ contains only a countable number of bounded parabolic points, we just have to prove that for such a point ξ, we have $\mu(\{\xi\}) = 0$.

Let ξ be the fixed point of some maximal parabolic subgroup $\mathcal{P} = \{p_k,\ k\in\mathbb{N}\}$ of Γ, with $p_0 = id$. For any Borel set $V \subset \overline{\Omega}$ containing an open neighbourhood of ξ in $\partial\Omega$, we have

$$\mu(\{\xi\}) \leqslant \mu(V) \leqslant \liminf_{s\to\delta_\Gamma^+} \mu^s(V).$$

So we just have to find a family of sets $(V_n)_{n\in\mathbb{N}}$ such that the right hand side goes to 0 when n goes to $+\infty$.
Choose an open fundamental domain $\mathcal{C} \in \Omega$ containing o. We let $V_n = \bigcup_{k\geqslant n} p_k(\overline{\mathcal{C}})$, such that each V_n contains an open neighbourhood of ξ in $\partial\Omega$. We have, for $s > \delta_\Gamma$,

$$\mu^s(V_n) = \frac{1}{g'_\Gamma(s,o)} \sum_{\gamma\in\Gamma} h(d_\Omega(o,\gamma o))e^{-sd_\Omega(o,\gamma o)}\mathbf{1}_{V_n}(\gamma o).$$

Let $\Gamma' = \{g\in\Gamma, go\in\overline{\mathcal{C}}\}$ be the subset of elements of Γ that do not move o outside $\overline{\mathcal{C}}$. Then

$$\mu^s(V_n) = \frac{1}{g'_\Gamma(s,o)} \sum_{k\geqslant n}\sum_{\gamma\in\Gamma'} h(d_\Omega(o,p_k\gamma o))e^{-sd_\Omega(o,p_k\gamma o)}.$$

Now remark that

$$d_\Omega(o,p_k\gamma o) = d_\Omega(o,\gamma o) + d_\Omega(o,p_k o) - 2(\gamma o|p_k^{-1}o)_o.$$

Nothing depends on the choice of o, and we can take it inside $C(\Lambda_\Gamma)$. Theorem 1.4.8 implies that the horoball $H_0 = \{y\in\Omega,\ b_\xi(o,y)\leqslant 0\}$ contains only a finite number of translates of Ω, so we can assume that H_0 only contains o, which actually would lie on the boundary of H_0. All other γo's, $\gamma\in\Gamma'$, lie at a distance at least $d>0$ from H_0. Thus, we can find some $r>0$ such that for $n>n_0$ large enough, V_n is contained in every lightcone $\mathcal{F}_r(\gamma o,o)$ based at γo, for $\gamma\in\Gamma'$. Lemma 1.2.1 now implies that $(go|p_k^{-1}o)_o \leqslant r$. From that and the fact that h is increasing, we get

$$\mu^s(V_n) = \frac{e^{2sr}}{g'_\Gamma(s,o)} \sum_{k\geqslant n} e^{-sd_\Omega(o,p_k o)} \sum_{\gamma\in\Gamma'} h(d_\Omega(o,p_k o)+d_\Omega(o,\gamma o))e^{-sd_\Omega(o,\gamma o)}.$$

Let $\eta>0$ and $t_\eta>0$ such that $\delta_\Gamma-\eta>\frac{1}{2}$, and for $t>t_\eta$, $h(t+s)\leqslant h(t)e^{\eta s}$. Only a finite number of $\gamma\in\Gamma'$ are such that $d(o,\gamma o)\leqslant t_\eta$; call G the set of such elements. Thus,

$$\mu^s(V_n) \leqslant \frac{e^{2sr}}{g'_\Gamma(s,o)} \sum_{k\geqslant n} e^{-sd_\Omega(o,p_k o)} \quad \Big(\sum_{\gamma\in G} h(d_\Omega(o,p_k o)+d_\Omega(o,\gamma o))e^{-sd_\Omega(o,\gamma o)}$$

$$+ \sum_{\gamma\in\Gamma'\smallsetminus G} h(d_\Omega(o,\gamma o))e^{-sd_\Omega(o,\gamma o)}e^{\eta d_\Omega(o,p_k o)}\Big).$$

The sum over G is obviously bounded by some constant C independent of s, so that

$$\mu^s(V_n) \leqslant \frac{e^{2sr}}{g'_\Gamma(s,o)} \left(C \sum_{k\geqslant n} e^{-sd_\Omega(o,p_k o)} + \left(\sum_{k\geqslant n} e^{-(s-\eta)d_\Omega(o,p_k o)} \right) \left(\sum_{g\in\Gamma'} h(d_\Omega(o,go)) e^{-sd_\Omega(o,go)} \right) \right)$$

Since $\delta_\Gamma - \eta > \frac{1}{2}$, the sum

$$\sum_{k\geqslant 0} e^{-(\delta_\Gamma-\eta)d(o,p_k o)} \tag{4.5}$$

converges. By letting $s \searrow \delta_\Gamma$, we get

$$\mu(V_n) \leqslant e^{2\delta r} \left(\sum_{k\geqslant n} e^{-(\delta_\Gamma-\eta)d_\Omega(o,p_k o)} \right) \mu(\partial\Omega).$$

The convergence in (4.5) implies that the right hand side goes to 0 when n goes to $+\infty$, proving that

$$\mu(\{\xi\}) = 0.$$

$\square$

Before completing the proof of theorem 4.3, note that this already implies the

Corollary 4.3.6. *Let $M = \Omega/\Gamma$ be a geometrically finite surface. Then Γ is divergent.*

Proof. The last proposition implies that Λ_r has full μ_x-measure, for any Patterson-Sullivan measure μ_x. Theorem 4.2.4 now gives that Γ is divergent. $\square$

Proof of theorem 4.3. Let μ_{BM} be a Bowen-Margulis measure on HM. Call $\tilde{\mu}_{BM}$ its lift to $H\Omega$ and μ the associated Γ-invariant measure on $\partial^2\Omega$. μ_{BM} is supported on the nonwandering set, which is contained in the homogeneous bundle $HC(M)$ over $C(M)$. Theorem 1.4.8 provides a decomposition of $C(M)$ into a compact part and a finite number of cusps C_i, $1 \leqslant i \leqslant p$. Each C_i is a quotient $C(\Lambda_\Gamma) \cap \Gamma.H/\Gamma$, where H is a horoball based at a fixed point of a corresponding maximal parabolic subgroup of Γ. So, we just have to prove that $\mu_{BM}(HC_i)$ is finite.
So, let $\mathcal{P}$ be a maximal parabolic subgroup of Γ and C be an open fundamental domain for $\mathcal{P}$ on Ω. We want to prove that $\tilde{\mu}_{BM}(H(C\cap H))$ is finite. The intersection $D = \partial C \cap \Lambda_\Gamma \smallsetminus \{p\}$ is a compact fundamental domain for the action of $\mathcal{P}$ on $\Lambda_\Gamma \smallsetminus \{p\}$ and we have, from the description made in the proof of theorem 4.1.2,

$$\begin{aligned}
\tilde{\mu}_{BM}(H(C\cap H)) &= \int_{\partial^2\Omega} l((\xi^-\xi^+)\cap(C\cap H))\ d\mu(\xi^-,\xi^+) \\
&= \sum_{p,q\in\mathcal{P}} \int_{pD\times qD} l((\xi^-\xi^+)\cap(C\cap H))\ d\mu(\xi^-,\xi^+) \\
&= \sum_{p,q\in\mathcal{P}} \int_{D\times p^{-1}qD} l((\xi^-\xi^+)\cap p^{-1}(C\cap H))\ d\mu(\xi^-,\xi^+) \\
&= \sum_{p\in\mathcal{P}} \int_{D\times pD} l((\xi^-\xi^+)\cap H)\ e^{-2\delta_\Gamma(\xi^+|\xi^-)_o} d\mu_o^2(\xi^-,\xi^+).
\end{aligned}$$

Since D is compact, we can find $r > 0$ such that any geodesic emanating from D and passing through H intersects $B(o, r)$. Now if $(\xi^- \xi^+)$ is such a geodesic with $\xi^+ \in pD$ then $(\xi^- \xi^+)$ also intersects $pB(o, r) = B(po, r)$. From that we deduce that

$$l((\xi^- \xi^+) \cap H) \leqslant d_\Omega(o, po) + r.$$

Furthermore, $pD \subset \mathcal{O}_r(o, po)$ and Sullivan's shadow lemma 4.2.2 implies

$$\mu_o(pD) \leqslant C_o e^{-\delta_\Gamma d_\Omega(o,po)}.$$

Thus

$$\tilde{\mu}_{BM}(H(C \cap H)) \leqslant C_o \sum_{p \in \mathcal{P}} (d_\Omega(o, po) + r) e^{-\delta_\Gamma d_\Omega(o,po)}.$$

Since $\delta_\mathcal{P} < \delta_\Gamma$, this series converges. □

4.4 Volume entropy and critical exponent for finite volume surfaces

The aim of this section is to prove that, on a surface of finite volume, volume entropy and critical exponent coincide, generalizing what is a trivial observation for a compact manifold.

Theorem 4.4.1. *Let $M = \Omega/\Gamma$ be a surface of finite volume. Then $h_{vol} = \delta_\Gamma$.*

(This result in true in all dimensions, and there is a similar result for geometrically finite quotients; the proof is the same, see [27].)

The proof of this result is the one given in [29], where the authors study manifolds of pinched negative curvature. They prove that the equality $h_{vol} = \delta_\Gamma$ always holds if the manifold is asymptotically 1/4-pinched, that is, the curvature in the cusps tend to be 1/4-pinched. They also construct examples whose curvature is arbitrarily close to being 1/4-pinched, but where equality fails.
Once again, the essential problem is to understand the behaviour of parabolic groups. In our case, some parts of the proof of the equality are really simplified by the transparence of the geometry. However, we also need specific results to overpass the non-Riemannian nature of the metric: these are contained in lemmas 4.4.3 and 4.4.4. But first, we need to recall the

Proposition 4.4.2 (L. Marquis, lemme 7.10 in [56])**.** *If $\Omega \subset \mathbb{RP}^2$ admits a quotient of finite volume, then (Ω, d_Ω) is Gromov-hyperbolic.*

Recall that for a discrete subgroup G of $Isom(\Omega, d_\Omega)$,

$$N_G(x, R) = \sharp\{g \in G,\ d_\Omega(x, gx) \leqslant R\}$$

denotes the number of elements g of G such that $gx \in B(x, R)$.

Lemma 4.4.3. *Let $C > 1$ be arbitrarily close to 1 and $\mathcal{P}$ a discrete parabolic subgroup of $Isom(\Omega, d_\Omega)$ fixing $p \in \partial\Omega$. Then, for any sufficiently small horoball H based at p and any $x \in \partial H$, there exists $D > 1$ such that*

$$\frac{1}{D} N_\mathcal{P}(x, \frac{R}{C}) \leqslant vol(B(x, R) \cap H) \leqslant D N_\mathcal{P}(x, CR).$$

Proof. It is known (see [29] for example), that in the hyperbolic space, we have, for any maximal parabolic subgroup $\mathcal{P}$, any horoball H fixed by $\mathcal{P}$ and any point $x \in \partial H$,

$$vol(B(x,R) \cap H) \asymp N_{\mathcal{P}}(x,R). \tag{4.6}$$

Now, we know from corollary 1.4.9 that, on any sufficiently small horoball H based at the fixed point p of $\mathcal{P}$, we can find two $\mathcal{P}$-invariant hyperbolic metrics $\mathtt{h}$ and $\mathtt{h}'$ such that

$$\frac{1}{C}\mathtt{h}' \leqslant \mathtt{h} \leqslant F \leqslant \mathtt{h}' \leqslant C\mathtt{h}.$$

So take such a small horoball H and pick $x \in \partial H$. We have for any $R > 0$,

$$B_{\mathtt{h}'}(x, \frac{R}{C}) \subset B_{\mathtt{h}}(x,R) \subset B(x,R) \subset B_{\mathtt{h}'}(x,R) \subset B_{\mathtt{h}}(x,CR),$$

where $B_{\mathtt{h}}$ and $B_{\mathtt{h}'}$ denote metric balls for $\mathtt{h}$ and $\mathtt{h}'$. If we denote by $vol_{\mathtt{h}}$ and $vol_{\mathtt{h}'}$ the Riemannian volumes associated to $\mathtt{h}$ and $\mathtt{h}'$, we have

$$vol_{\mathtt{h}'} \leqslant vol \leqslant vol_{\mathtt{h}}.$$

Hence

$$vol_{\mathtt{h}'}(B_{\mathtt{h}'}(x, \frac{R}{C}) \cap H) \leqslant vol(B(x,R) \cap H) \leqslant vol_{\mathtt{h}}(B_{\mathtt{h}}(x,CR) \cap H).$$

Now equation (4.6) provides a real $D > 1$ such that

$$\frac{1}{D} N_{\mathcal{P}}^{\mathtt{h}'}(x, \frac{R}{C}) \leqslant vol(B(x,R) \cap H) \leqslant D N_{\mathcal{P}}^{\mathtt{h}}(x,CR),$$

where $N_{\mathcal{P}}^{\mathtt{h}}(x,R)$ is the number of points of the orbit $\mathcal{P}.x$ in the ball of radius R for $\mathtt{h}$; the same for $\mathtt{h}'$.
Well, of course, the horoballs involved in equation (4.6) are the hyperbolic horoballs, and not those for F, so we have to be a bit more cautious. But if $\mathcal{H}_{\mathtt{h}}$ is the horosphere for $\mathtt{h}$ based at p and passing through x, then the maximal $\mathtt{h}$-distance between $\mathcal{H}$ and $\mathcal{H}_{\mathtt{h}}$ is finite, because $\mathcal{P}$ acts cocompactly on $\mathcal{H} \smallsetminus \{p\}$ and $\mathcal{H}_{\mathtt{h}} \smallsetminus \{p\}$. Hence, there exists some $D' > 0$ such that, for any $R > 0$,

$$|vol_{\mathtt{h}}(B_{\mathtt{h}}(x,R) \cap H) - vol_{\mathtt{h}}(B_{\mathtt{h}}(x,R) \cap H_{\mathtt{h}})| \leqslant D' N_{\mathcal{P}}(x,R),$$

where $H_{\mathtt{h}}$ is the horoball defined by $\mathcal{H}_{\mathtt{h}}$. Hence the claim that such a D exists.

We can conclude by remarking that, since $\mathtt{h} \leqslant F \leqslant \mathtt{h}'$, we have

$$N_{\mathcal{P}}^{\mathtt{h}}(x,R) \leqslant N_{\mathcal{P}}(x,R) \leqslant N_{\mathcal{P}}^{\mathtt{h}'}(x,C).$$

□

Proof of theorem 4.4.1. We already know that $\delta_\Gamma \leqslant h_{vol}$, so we only have to prove the converse.

Fix $C > 1$ arbitrarily close to 1, and pick $o \in \Omega$. Choose a fundamental domain for the action of Γ on Ω, that contains o, and decompose it into

$$C_0 \bigsqcup \sqcup_{i=1}^{l} C_i,$$

where C_0 is compact and the $C_i,\ 1 \leqslant i \leqslant l$, are cusps, based at $\xi_i \in \partial\Omega$. Each C_i is the fundamental domain for the action of a maximal parabolic subgroup $\mathcal{P}_i$ on the horoball H_{ξ_i} based at ξ_i. We assume that the C_i are chosen small enough so that the horoballs H_{ξ_i} satisfy lemma 4.4.3, with the constant C that was chosen.

The ball $B(o,R)$ of radius $R \geqslant 0$ can then be decomposed into

$$B(o,R) = (\Gamma.C_0 \cap B(o,R)) \sqcup \left(\sqcup_{i=1}^{l} \Gamma.H_{\xi_i} \cap B(o,R)\right),$$

so that

$$vol(B(o,R)) = vol(\Gamma.C_0 \cap B(o,R)) + \sum_{i=1}^{l} vol(\Gamma.H_{\xi_i} \cap B(o,R)).$$

For the first term we have $vol(\Gamma.C_0 \cap B(o,R)) \leqslant N_\Gamma(o,R) vol(C_0)$. Let us study the second one.

For each horoball $H_{\gamma\xi_i} = \gamma H_{\xi_i}$, denote by $x_{\gamma,i}$ the intersection of $(o\gamma\xi_i)$ with $\partial H_{\gamma\xi_i}$, that is the projection of o on $H_{\gamma\xi_i}$. For any $\gamma \in \Gamma$, we denote by $\overline{\gamma} \in \Gamma$ one of the elements $g \in \Gamma$ such that $x_{\gamma,i} \in g.C_i$, whose number is finite; it is the "first element for which $H_{\gamma\xi_i}$ intersects $B(o,R)$". Let $\overline{\Gamma}$ be the set of such elements.

The main remark is the following lemma, which is a classical one in pinched negative curvature: for each $\theta \in (0,\pi)$, there exists a constant $C(\theta)$ such that, for any geodesic triangle xyz whose angle at y is at least θ, the path $x \to y \to z$ on the triangle is a quasi-geodesic between x and z with an error at most $C(\theta)$.

Lemma 4.4.4. *There exists $r > 0$ such that, for any $\gamma \in \Gamma,\ 1 \leqslant i \leqslant l$ and $z \in H_{\gamma\xi_i}$, the path consisting of the segments $[ox_{\gamma,i}]$ and $[x_{\gamma,i}z]$ is a quasi-geodesic with an error of at most r, that is,*

$$d_\Omega(o,z) \geqslant d_\Omega(o,x_{\gamma,i}) + d_\Omega(x_{\gamma,i},z) - r.$$

Proof. Take $\gamma \in \Gamma,\ 1 \leqslant i \leqslant l$ and $z \in H_{\gamma\xi_i}$. Since (Ω, d_Ω) is Gromov-hyperbolic (proposition 4.4.2), there is some $\delta \geqslant 0$ such that every triangle is δ-thin. So there exists $p \in [oz]$, such that

$$d_\Omega(p,[x_{\gamma,i}z]) \leqslant \delta,\ d_\Omega(p,[ox_{\gamma,i}]) \leqslant \delta.$$

Hence, we can find points $o' \in [ox_{\gamma,i}]$ and $z' \in [x_{\gamma,i}z]$, such that

$$d_\Omega(o',p) + d_\Omega(p,z') \leqslant 2\delta.$$

By the triangular inequality, the distance between o' and z' is then less than 2δ. By convexity of the metric balls and the horospheres, we get that $x_{\gamma,i} \in B(o',2\delta)$, so that

$$d_\Omega(o',x_{\gamma,i}) + d_\Omega(x_{\gamma,i},z') \leqslant 4\delta.$$

That gives

$$\begin{aligned} d_\Omega(o,x_{\gamma,i}) + d_\Omega(x_{\gamma,i},z) &\leqslant d_\Omega(o,o') + d_\Omega(o',x_{\gamma,i}) + d_\Omega(x_{\gamma,i},z') + d_\Omega(z',z) \\ &\leqslant 4\delta + d_\Omega(o,p) + d_\Omega(p,o') + d_\Omega(z',p) + d_\Omega(p,z) \\ &\leqslant 6\delta + d_\Omega(o,z). \end{aligned}$$

□

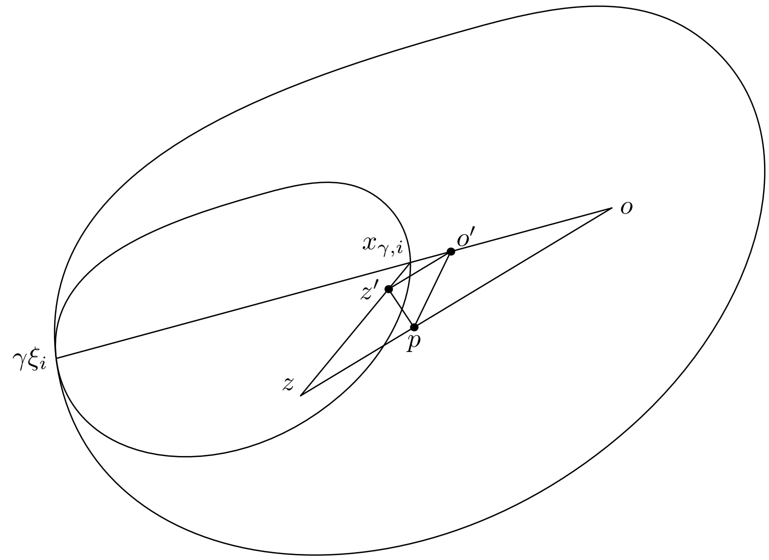

Figure 4.1: Quasi-geodesics

Now, if z is a point in $\gamma.H_{\xi_i} \cap B(o,R)$, for some $\gamma \in \Gamma$, $1 \leqslant i \leqslant l$ and $R > 0$, this lemma implies that

$$d_\Omega(o, x_{\gamma,i}) + d_\Omega(x_{\gamma,i}, z) \leqslant d_\Omega(o,z) + r \leqslant R + r.$$

But there exists $c \geqslant 0$, so that $d_\Omega(o, x_{\gamma,i}) \geqslant d_\Omega(o, \overline{\gamma}o) - c$: take for c the maximal distance between o and the boundary $\partial C_i \cap \partial H_{\xi_i} \smallsetminus \{\xi_i\}$. Then

$$d_\Omega(x_{\gamma,i}, z) \leqslant R + r - d_\Omega(o, \overline{\gamma}o) + c.$$

Let $K = r + c$. For any $\gamma \in \Gamma$, $1 \leqslant i \leqslant l$, and $R > 0$, we thus have

$$\gamma.H_{\xi_i} \cap B(o,R) \subset \gamma.H_{\xi_i} \cap B(x_{\gamma,i}, R - d(o, \overline{\gamma}o) + K).$$

This gives an efficient way to evaluate $vol(\Gamma.H_{\xi_i} \cap B(o,R))$. Indeed,

$$\begin{aligned}
vol(\Gamma.H_{\xi_i} \cap B(o,R)) &= \sum_{\overline{\gamma}\in\overline{\Gamma}} vol(\overline{\gamma}.H_{\xi_i} \cap B(o,R)) \\
&\leqslant \sum_{\overline{\gamma}\in\overline{\Gamma}} vol(\overline{\gamma}.H_{\xi_i} \cap B(x_{\gamma,i}, R - d(o,\overline{\gamma}o) + K)) \\
&\leqslant \sum_{0\leqslant n\leqslant [R]} \sum_{\substack{\overline{\gamma} \in \overline{\Gamma} \\ n \leqslant d_\Omega(o,\overline{\gamma}o) \leqslant n+1}} vol(\overline{\gamma}.H_{\xi_i} \cap B(x_{\gamma,i}, R - n - 1 + K)) \\
&\leqslant \sum_{0\leqslant k\leqslant [R]} N_{\overline{\Gamma}}(o,k,k+1) vol(H_{\xi_i} \cap B(x_i, R - k + K)),
\end{aligned}$$

where $x_i = x_{Id,i}$ and, for any subset S of Γ and $0 \leqslant r < R$,

$$N_S(o,r,R) = \sharp\{\gamma \in S,\ r \leqslant d_\Omega(o,\gamma o) < R\}.$$

Lemma 4.4.3 gives

$$vol(\Gamma.H_{\xi_i} \cap B(o,R)) \leqslant D \sum_{0\leqslant k\leqslant [R]} N_{\overline{\Gamma}}(x_i,k,k+1) N_{\mathcal{P}_i}(x_i, C(R-k)) \tag{4.7}$$

for some $D > 1$ that can be chosen independent of i. Furthermore, since the critical exponent of each $\mathcal{P}_i$ is $\frac{1}{2}$, there exists $M \geqslant 1$, independent of i but depending on C, such that

$$\frac{1}{M} e^{(\frac{1}{2}-(C-1))R} \leqslant N_{\mathcal{P}_i}(x_i,R)) \leqslant M e^{\frac{1}{2}R}.$$

(There is no need of a corrective term for the upper bound from the second point of corollary 4.2.3.) Hence,

$$N_{\mathcal{P}_i}(x_i,CR) \leqslant M e^{\frac{1}{2}CR} \leqslant M e^{(\frac{1}{2}-(C-1))CR}\ e^{C(C-1)R} \leqslant M^2 e^{C(C-1)R} N_{\mathcal{P}_i}(x_i,R)).$$

With (4.7), that implies

$$vol(\Gamma.H_{\xi_i} \cap B(o,R)) \leqslant DM^2 e^{C(C-1)R} \sum_{0\leqslant k\leqslant [R]} N_{\overline{\Gamma}}(x_i,k,k+1) N_{\mathcal{P}_i}(x_i,R-k). \tag{4.8}$$

Finally, remark that any $\gamma \in \Gamma$ such that $d_\Omega(x_i,\gamma x_i) < R$ can be written in a unique way as $\gamma = \overline{\gamma_i} p_i$, with $d_\Omega(x_i,\overline{\gamma_i}x_i) < R$ and $p_i \in \mathcal{P}_i$ so that

$$d(x_i,p_i x_i) + d_\Omega(x_i,\overline{\gamma_i}x_i) \geqslant R.$$

Hence

$$N_\Gamma(x_i,R) \geqslant \sum_{0\leqslant k\leqslant [R]} N_{\overline{\Gamma}}(x_i,k,k+1) N_{\mathcal{P}_i}(x_i,R-k). \tag{4.9}$$

(4.8) and (4.9) together yield

$$vol(\Gamma.H_{\xi_i} \cap B(o,R)) \leqslant DM^2 e^{C(C-1)R} N_\Gamma(x_i,R),$$

so that, putting everything together,

$$vol(B(o,R)) \leqslant N e^{C(C-1)R} N_\Gamma(o,R),$$

for some constant $N > 1$. That gives

$$h_{vol} \leqslant \delta_\Gamma + C(C-1).$$

Since C can be chosen arbitrarily close to 1, that yields

$$h_{vol} \leqslant \delta_\Gamma.$$

□

Chapter 5

Entropies

This last chapter proves the existence and uniqueness of a measure of maximal entropy for some specific quotients. It extends Ruelle inequality and its case of equality to noncompact quotients of Gromov-hyperbolic Hilbert geometries. An entropy rigidity theorem is then proved in the case of compact quotients and finite volume surfaces.

5.1 The measure of maximal entropy

The goal of this part is to prove the following theorem.

Theorem 5.1.1. *Let $M = \Omega/\Gamma$ be the quotient manifold of a strictly convex proper open set $\Omega \subset \mathbb{RP}^n$ with C^1 boundary by a nonelementary group $\Gamma \subset Isom(\Omega, d_\Omega)$. Assume there exists a finite Bowen-Margulis measure and denote by μ_{BM} the probability one. If the geodesic flow has no zero Lyapunov exponent on the nonwandering set, then μ_{BM} is the unique measure of maximal entropy and*

$$h_{top} = h_{\mu_{BM}} = \delta_\Gamma.$$

Since the geodesic flow on a geometrically finite surface has been proved to be uniformly hyperbolic on the nonwandering set (theorem 2.5.2), it has no zero Lyapunov exponent. Furthermore, theorem 4.3.1 claims that there exists a finite Bowen-Margulis measure, and the theorem admits the following

Corollary 5.1.2. *Let $M = \Omega/\Gamma$ be a geometrically finite surface and μ_{BM} its probability Bowen-Margulis measure. Then μ_{BM} is the unique measure of maximal entropy and*

$$h_{top} = h_{\mu_{BM}} = \delta_\Gamma.$$

A more general version of this theorem, including the cases for which there is no finite Bowen-Margulis measure, was proved for quotients of Hadamard manifolds of pinched negative curvature by Otal and Peigné [61]. They actually proved that, if there is no finite Bowen-Margulis measure, then we still have $h_{top} = \delta_\Gamma$ but there is no measure of maximal entropy. Obviously, the assumption of no zero Lyapunov exponent is useless in pinched negative curvature.
Such a version is probably true in our setting. Nevertheless, no example of such more exotic quotient is known so far for Hilbert geometry, and we decided to restrict ourselves to the currently more relevant cases. The assumption of no zero Lyapunov exponent can be seen as a counterpart of

pinched negative curvature. Anyway, I have no idea if there can exist a quotient with zero Lyapunov exponent on the nonwandering set.

The proof of the theorem follows the one given by Otal and Peigné, but it is simplified. I had the opportunity to follow a mini-course given by François Ledrappier about this result; it was really helpful to understand the whole strategy and most of the simplifications come from what I learnt either from this lecture or from François himself.

The idea is a classical one and comes from the pioneering works of Ledrappier, Pesin, Strelcyn and Young. This is based on Rokhlin theory of measurable partitions. Let us explain here the strategy. There are three things to prove (see section 5.1.4):

- for any invariant probability measure μ, $h_\mu \leqslant h_{\mu_{BM}}$;
- the equality $h_\mu = h_{\mu_{BM}}$ implies that $\mu = \mu_{BM}$;
- $h_{\mu_{BM}} = \delta_\Gamma$.

To prove these three points, given a measure μ, we construct a well-adapted partition which allows us to compute the entropy of μ. These are measurable partitions, as introduced by Rokhlin, which are subordinate to the unstable foliation, that is, its atoms are open pieces of unstable manifolds. Section 5.1.2 explains how to construct such partitions, while the next one proves that such a partition α gives all the entropy, that is $h_\mu = h_\mu(\alpha, \varphi)$. The proof that it gives all the entropy relies on a construction of Mañé and lemma 5.1.5, that was indicated by François Ledrappier in his lecture. The use of this lemma really simplifies the proof given by Otal and Peigné, who instead had used a more general and complicated argument that would also work in the presence of zero Lyapunov exponents.
Since the partition consists of open pieces of unstable manifolds, it gives an efficient way of computing the entropy of μ_{BM}, because we know how the flow acts on the Margulis measures. It also allows us to compare the entropy of μ_{BM} with the entropy of another measure μ, and prove the first two points.

Note that most of the tools should work in the case there would be some zero Lyapunov exponent. It is still possible to construct a measurable partition that gives all the entropy. This partition would be subordinate to the W_1^u-manifold, corresponding to the smallest positive Lyapunov exponent, and to prove it gives all the entropy, we should use the more complicated argument given by Otal and Peigné. The problem would arrive later: the W_1^u-manifolds are submanifolds of positive codimension of the unstable manifolds, and we do not know how the flow acts on the conditional measures of μ_{BM} on W_1^u-manifolds. Thus, it is not clear this partition can help to compute the entropy. However, since we do not know if there exist quotients with zero Lyapunov exponents, trying to prove something in this case is not currently relevant.

5.1.1 Measurable partitions

We know from Rokhlin theorem 4.1.4 that, given an invariant probability measure, there always exists a countable partition, which gives all the entropy. But we dot know how this partition looks like, and it does not help to effectively compute the entropy of the measure. For this, we will use more general partitions that were introduced by Rokhlin in [68] (see also [69] and [62] for more modern presentations). We recall here the most important facts about these partitions.

A partition α of a probability space $(X, \mathcal{A}, \mu)$ is a collection $(\alpha_i)_{i\in I}$ of measurable subsets of X such that

$$\mu(\alpha_i \cap \alpha_j) = 0, \quad \mu(X \setminus \cup_{i\in I}\alpha_i) = 0.$$

We say that a partition α is finer than β, and write $\alpha \succ \beta$ or $\beta \prec \alpha$, if any atom α_i is a subset of some atom β_j. If α and β are two partitions, the joint partition $\alpha \vee \beta$ is defined as

$$\alpha \vee \beta = \{A \cap B, \ A \in \alpha, \ B \in \beta\}.$$

The joint partition $\alpha \vee \beta$ refines α and β. If $\alpha \succ \beta$, then $\alpha \vee \beta = \alpha$. The finest partition is the partition by points ϵ such that $\epsilon(x) = \{x\}$, and the least fine one is the trivial partition with one atom: X. To a partition α, we associate the quotient space X/α which consists of atoms of α. The projection $\pi_\alpha : X \longrightarrow X/\alpha$ is defined almost everywhere on X and is measurable since the atoms of the partition are measurable. We denote by $\overline{\mu}$ the measure $\pi_\alpha \mu$ on X/α.
A partition α is a **measurable partition** if there exists a family $(A_n)_{n\in\mathbb{N}}$ of measurable subsets such that $A = \cup_{n\in\mathbb{N}} A_n$ has full measure and, for any two atoms α_i and α_j, there exists some n such that $A \cap \alpha_i \subset A_n$, $A \cap \alpha_j \subset A \setminus A_n$. Rokhlin proved that conditional measures with respect to a measurable partition can be defined, that is:

Theorem 5.1.3 (Rokhlin [68]). *Let α be a measurable partition. Then, to $\overline{\mu}$-almost every atom $a \in \alpha$, is associated a probability measure μ_a on X such that*

- *μ_a is supported on a;*
- *the application $x \longmapsto \mu_{\alpha(x)}$ is measurable;*
- *for any measurable set A, $\mu(A) = \int_{X/\alpha} \mu_a(A) d\overline{\mu}(a)$.*

The measure $\mu_{\alpha(x)}$ is called the conditional measure at x with respect to α.

The entropy of a measurable partition is defined by

$$H(\alpha) = -\int_X \log \ \mu(\alpha(x)) \ d\mu(x),$$

which generalizes definition 4.2. This definition is not interesting for those partitions whose atoms have measure zero, since their entropy is zero.

Consider an invertible transformation $\Phi : (X, \mu) \longrightarrow (X, \mu)$. The inversibility is not necessary for the definitions, but the tools and results are really different in the case of a noninvertible transformation. Since we want to apply it to our geodesic flow, there is no need of considering noninvertible transformations.

We want to define the entropy of a measurable partition α under Φ. Definition 4.3 would give zero for all those partitions whose atoms are negligible, thus another one is needed to take them into account.
Φ transforms the partition α in a new partition $\Phi\alpha$ whose atoms are the $\Phi^{-1}(\alpha_i), \ i \in I$. We say that a partition is increasing if $\Phi\alpha$ is finer than α, that is, $\Phi\alpha \succ \alpha$. That means that each atom α_i

is the union of atoms of $\Phi\alpha$. Thus it makes sense to consider the conditional entropy of $\Phi\alpha$ with respect to α given by

$$H(\Phi\alpha|\alpha) = \int_X \mu_{\alpha(x)}(\Phi\alpha(x))\ d\mu(x).$$

We then define the entropy of an **increasing** measurable partition by

$$h(\Phi, \alpha) = H(\Phi\alpha|\alpha)$$

(see section 4.1.2). If P is countable and increasing, then this definition coincide with the one given by (4.3).
Remark that, for any countable partition P, the partition $P^- = \vee_{i=-\infty}^{0} \Phi^i P$ is increasing, and we have

$$h(P, \Phi) = h(P^-, \Phi).$$

We thus have

$$h(\Phi) = \sup_\alpha h(\alpha, \Phi),$$

where the supremum is taken with respect to all measurable increasing partitions with finite entropy.

Of course, we can also do the same for **decreasing** partitions such that $\alpha \succ \Phi\alpha$; these are just increasing partitions for Φ^{-1}, that has the same entropy as Φ.

We say that a partition α is **generating** if

$$\bigvee_{i=-\infty}^{i=+\infty} \Phi^i \alpha = \epsilon$$

is the partition into points.

5.1.2 Leaf subordinated partitions

Let $M = \Omega/\Gamma$ be the quotient manifold of a strictly convex proper open set $\Omega \subset \mathbb{RP}^n$ with C^1 boundary by a nonelementary group $\Gamma \subset Isom(\Omega, d_\Omega)$. An ergodic measure is always supported on the nonwandering set. A general invariant probability measure can always be decomposed into a conservative and a dissipative part; the dissipative part does not change the entropy and the conservative part is supported on the nonwandering set. By decomposing the space into ergodic components, we can always assume that the measure is ergodic.

In what follows, we fix an ergodic probability measure m for the geodesic flow φ^t on HM, and we choose $T > 0$ such that $\Phi = \varphi^T$ is ergodic with respect to m. This is always possible, as claimed by lemma 7 in [61].

By Oseledets' theorem, m-almost every point in HM is regular with the same Lyapunov exponents. Assume m has no zero Lyapunov exponent, and call Λ_m the set of regular points with positive Lyapunov exponents $0 < \chi_1 < \cdots < \chi_p$, which is of full m-measure. At any point $w \in \Lambda_m$, for any vector $Z \in E^u(w) \smallsetminus \{0\}$,

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \overline{F}(d\varphi^t(Z)) \geqslant \chi_1.$$

We fix $0 < epsilon < \frac{\chi_1}{923}$. For any $w \in \Lambda_m$, there exists $C(w) > 0$ such that, for any $Z \in E^u(w)$ and $t \geqslant 0$,

$$\overline{F}(d\varphi^{-t}(Z)) \leqslant C(w)e^{-(\chi_1-\epsilon)t}\overline{F}(Z).$$

In fact, we can choose

$$C(w) = \sup\left\{\frac{\overline{F}(d\varphi^{-t}(Z))}{e^{-(\chi_1-\epsilon)t}\overline{F}(Z)},\ t \geqslant 0,\ Z \in E^u(w)\right\},$$

so that the function $C : \Lambda_m \longrightarrow (0, +\infty)$ is measurable and $C(\varphi^{-t}(w)) = O(1),\ t \to +\infty$.

Let $\Lambda_m(c) = C^{-1}((0, c))$ for $c > 0$. If $c' > c$, then $\Lambda_m(c') \supset \Lambda_m(c)$, and since $\Lambda_m = \bigcup_{c>0} \Lambda_m(c)$, there exists some $c_0 \geqslant 0$ such that, for any $c > c_0$, $m(\Lambda_m(c)) > 0$.

Theorem 5.1.4. *Let $M = \Omega/\Gamma$ be the quotient manifold of a strictly convex set Ω with C^1 boundary. Let m be an ergodic invariant measure on HM with no zero positive Lyapunov exponent. Then there exists a measurable, generating and decreasing partition α subordinate to the unstable foliation W^u.*

Such a partition will be called a W^u-partition with respect to m. By subordinate to W^u, we mean that $\overline{m}$-almost any atom of the partition α is an open subset of W^u.

We will need the concept of a **flow box**. For $w \in HM$ and $r > 0$, we denote by $W^s(w, r)$ (resp. $W^u(w, r)$) the r-neighbourhood of w in the stable manifold $W^s(w)$ (resp. unstable manifold $W^u(w)$), where distances are considered with respect to the metrics induced by the Finsler metric $\overline{F}$. The (closed) flow box $B_r(w_0)$ of size $r > 0$ (small enough) and origin $w_0 \in HM$ is

$$B_r(w_0) = \bigcup_{0\leqslant t\leqslant r} \varphi^t(B^{us}(w_0, r)),$$

where

$$B^{us}(w_0) = \{v \in W^u(w, r),\ w \in W^s(w_0, r)\}.$$

Obviously, r has to be chosen small enough so that all the images $\varphi^t(B^{us}(w_0, r))$ are disjoint for $0 \leqslant t \leqslant r$. By construction, $B_r(w_0)$ is foliated by the $\varphi^t(B^{us}(w_0, r)), 0 \leqslant t \leqslant r$, but also by pieces of unstable manifolds of diameter $2r$.

Proof of proposition 5.1.4. Take a $c > c_0$ such that $m(\Lambda_m(c)) > 0$. Consider a flow box $B_r := B_r(w_0)$ of size $r > 0$, with origin $w_0 \in \Lambda_m(c) \cap \operatorname{supp}(m)$, so that in particular $m(B_r \cap \Lambda_m(c)) > 0$. Define the partition α' of HM by B_r foliated by W^u-leaves, and $(B_r)^c$: if $w \in B_r$, the atom $\alpha'(w)$ is the connected component of w in $W^u(w) \cap B_r$; if $w \notin B_r$, then $\alpha'(w) = (B_r)^c$. Let

$$\alpha = \bigvee_{k=0}^{+\infty} \Phi^{-k}\alpha'.$$

This partition α is measurable, generating and decreasing.
We have to prove that for almost every $w \in HM$, the atom $\alpha(w)$ is an open neighbourhood of w in $W^u(w)$. For $k \in \mathbb{N}$, we have

$$\Phi^k\alpha'(w) = \Phi^{-k}(\alpha'(\Phi^k(w))),$$

hence

$$\alpha(w) = \bigcap_{k \in \mathbb{N}} \Phi^{-k}(\alpha'(\Phi^k(w))).$$

The interesting terms in this intersection are those when $\Phi^k(w) \in B_r$ since $\Phi^k(w)$ is then a piece of W^u-manifold. Since m is ergodic, almost any point $w \in HM$ will go through B_r infinitely often, so $\alpha(w)$ will be m-almost surely a piece of $W^u(w)$. Such a piece will be an open neighbourhood if every time w goes through B_r, it stays far enough from the boundary of B_r. More precisely, $\alpha(w)$ will be an open neighbourhood of w in $W^u(w)$ if there is no strictly increasing sequence of positive times $n_k,\ k \in \mathbb{N}$, such that

$$\lim_{k \to +\infty} d^u(\Phi^{-n_k}(w), \partial B_r) = 0,$$

where d^u denotes the metric generated by $\overline{F}$ on $W^u(w)$. (Remark that this metric is nothing else than the metric generated by the Hilbert metric F on the projection of $W^u(w)$ on M.)

But a classical Borel-Cantelli argument proves that this is true almost everywhere on any $\Lambda_m(c)$ for Lebesgue almost any $r > 0$ (see [2] p.285-288). Since $\Lambda_m = \cup_{n \in \mathbb{N}^*} \Lambda_m(n)$, the same holds on Λ_m. □

Lemma 5.1.5. *Let α be an increasing and generating m-measurable partition. If there exists some countable partition Q such that $Q^- \succ \alpha$, then*

$$h(\alpha, \Phi) \geqslant h(Q, \Phi)$$

Proof. We have

$$\begin{aligned} h(Q,\Phi) = H(\Phi Q|Q^-) \leqslant H(\Phi P|\alpha) \quad &= \lim_{n\to\infty} \frac{1}{n} H(P^{-n}|\alpha) \\ &\leqslant \lim_{n\to\infty} \frac{1}{n} H(P^{-n} \vee \Phi^n \alpha|\alpha) \\ &= \lim_{n\to\infty} \frac{1}{n}(H(P^{-n}|\Phi^n\alpha \vee \alpha) + H(\Phi^n\alpha|\alpha)) \\ &= h(\alpha, \Phi), \end{aligned}$$

since $\Phi^n \alpha \to \epsilon$. □

5.1.3 Mañé partitions

We here explain a construction of Mañé. This construction gives a finite partition P such that P^- is finer than the W^u-partition of theorem 5.1.4, that allows to apply lemma 5.1.5; see corollary 5.1.9.

We still assume that we have fixed an ergodic probability measure m for the geodesic flow φ^t on HM, and a time $T > 0$ such that $\Phi = \varphi^T$ is ergodic with respect to m.

For a relatively compact measurable subset B of positive m measure, the Mañé partition P_B induced by B will be the partition

$$P_B = B^c \bigsqcup \sqcup_{n \geqslant 1} B_n,$$

where $B_n = \{w \in B,\ \Phi^n w \in B,\ \Phi^i w \notin B,\ 0 < i < n\}$ for $n \geqslant 1$.

Lemma 5.1.6 (Kăc's lemma)**.** *If $P_B = B^c \bigsqcup \sqcup_{n\geqslant 1} B_n$ is the Mañé partition induced by B then*

$$\sum nm(B_n) = 1.$$

Proof. Since m is ergodic, we have

$$HM = \bigsqcup_{n\geqslant 1} \sqcup_{0\leqslant i\leqslant n-1} \Phi^i B_n \ mod\ 0,$$

and by invariance of m,

$$1 = \sum_n \sum_{0\leqslant i\leqslant n-1} m(\Phi^i B_n) = \sum nm(B_n).$$

The next lemma tells us that P_B has then finite entropy. □

Lemma 5.1.7. *If $(x_n) \in [0,1]^{\mathbb{N}}$ is such that $\sum_n nx_n < +\infty$, then*

$$-\sum x_n \log x_n < +\infty$$

Proof. This is lemma 10.5.3 in [2], p.316. □

We keep using the notations of the preceding section. We choose a $c > 0$ such that $m(\Lambda_m(c)) > 0$. Consider a closed flow box $B'_r := B'_r(w_0)$ of size $r > 0$, with origin $w_0 \in \Lambda_m(c) \cap \operatorname{supp}(m)$. Consider $B_r = \cup_{w\in B'_r\cap\Lambda_m}(W^u(w) \cap B'_r)$ and the Mañé partition

$$B = P_{B_r} = B_r^c \bigsqcup \sqcup_{n\geqslant 1} B_n$$

induced by B_r.

We refine this partition in the following way: cut B_n into K_n pieces $(B_{n,k})_{1\leqslant k\leqslant K_n}$ such that each $\Phi^n(B_{n,k})$ is exactly one connected component of $\Phi^n(B_n) \cap B_r$. The number K_n of pieces can be chosen smaller than $Ce^{(\chi_1+\epsilon)n}$ for some $C > 0$. Now refine the partition B into B' by cutting B_r into

$$B_r = \bigsqcup_n \bigsqcup_{1\leqslant k\leqslant K_n} B_{n,k}.$$

Finally, recall from the construction of the flow box that $B^{us}(w_0)$ denotes "the basis of the box". Let

$$C_i = \bigcup_{r/2^{i+1}<t\leqslant r/2^i} \varphi^t(B^{us}(w_0))$$

for $i \geqslant 0$, and consider the partition C whose atoms are the $C_i,\ i \geqslant 0$ and $(\cup_{i\geqslant 0} C_i)^c = (B'_r)^c$. Each C_i has positive measure, since $w_0 \in \operatorname{supp}(m)$ and C has clearly finite entropy: if $M = m(C_0)$ then $m(C_i) = \frac{M}{2^i}$ and

$$H(C) = \sum_{i\geqslant 0} \frac{M}{2^i} \log \frac{2^i}{M} < +\infty.$$

Let $Q = C \vee B'$.

Proposition 5.1.8. *Assume m has no zero Lyapunov exponent. Then for Lebesgue-almost all r small enough, Q is generating and $P = Q^-$ is a subpartition of the W^u-partition α induced by B_r.*

Proof. First check that Q has finite entropy: we have

$$H(Q) \leqslant H(C) + H(B') + H(Q|B')$$

and

$$\begin{aligned} H(Q|B') &\leqslant -\sum_n \sum_{1\leqslant k\leqslant K_n} m(B_{n,k}) \log \frac{m(B_{n,k})}{m(B_n)} \\ &\leqslant -\sum_n m(B_n) \sum_{1\leqslant k\leqslant K_n} \frac{1}{K_n} \log \frac{1}{K_n} \\ &\leqslant D \sum_n nm(B_n) \\ &= D \\ &< +\infty, \end{aligned}$$

from Kăc's lemma.

Now, we prove that for almost all w, $P(w) \subset W^u(w,r)$ and thus P is generating.
For two points v and w, we have $v \in P(w)$ if for any $n \geqslant 0$, $\Phi^{-n} v \in Q(\Phi^{-n} w)$. In particular, the preimages of v and w are in C_i at the same moment. Let $0 < n_1(v) < \cdots < n_i(v) < \cdots$ be the times for which $\Phi^{-n_k} v \in C_i$; since $m(C_i) > 0$, the set $N(v) = \{n_k(v)\} \subset \mathbb{N}$ is infinite for almost every point $v \in HM$, and

$$P(v) \subset \bigcap_{i\geqslant 1} \Phi^{n_i}(C_i) \subset W^u(v).$$

Thus there exists a smallest $N \geqslant 0$ such that $\Phi^{-N} v \in B_r$ and, for any $n \geqslant N$, $\Phi^{-n} v \in W^u(\Phi^{-n} w, r)$. But $\Phi^{-N} v$ and $\Phi^{-N} w$ are both in some $B_{p,k}$, so that $\Phi^{-N+p} w$ and $\Phi^{-N+p} v$ are in

$$W^u(\Phi^{-N+p} w) \cap \Phi^{-N+p}(B_{p,k}) \subset W^u(\Phi^{-N+p} w, r).$$

Since N is the smallest positive number for which this may occur, we have $-N + p > 0$, hence for any $-N \leqslant i \leqslant -N+p$, $\Phi^{-i} v \in W^u(w,r)$. In particular, $v \in W^u(w,r)$, that is $P(w) \subset W^u(w,r)$.

It is clear from the construction that $P^- \succ \alpha$.

□

Corollary 5.1.9. *Let $M = \Omega/\Gamma$ be the quotient manifold of a strictly convex proper open set Ω with C^1 boundary. If an invariant ergodic measure m on HM has no zero Lyapunov exponent and α is the W^u-partition induced by B_r, then*

$$h(\Phi) = h(\Phi, \alpha).$$

Proof. Last proposition tells us that there exists a generating countable partition Q such that $Q^- \succ \alpha$. Kolmogorov-Sinai theorem gives $h(\Phi) = h(\Phi, Q)$ and lemma 5.1.5 yields $h(\Phi, \alpha) \geqslant h(\Phi, Q)$.
□

5.1.4 Proof of theorem 5.1.1

The following lemma is general and will be used a couple of times. This is lemma 8 in [61] and we omit the proof.

Lemma 5.1.10. *Let $f : HM \longmapsto \mathbb{R}$ be a measurable function such that $f \circ \Phi - f$ has its negative part in $L^1(m)$. Then*

$$\int f \circ \Phi - f \; dm = 0.$$

Let us first prove an intermediate

Proposition 5.1.11. *Let $M = \Omega/\Gamma$. Assume there exists a finite Bowen-Margulis measure and denote by μ_{BM} the probability one. If μ_{BM} has no zero Lyapunov exponent, then*

$$h_{\mu_{BM}} = \delta_\Gamma.$$

Proof. Let us abbreviate μ_{BM} by μ. Let α be a W^u-partition for μ as in theorem 5.1.4. We have from corollary 5.1.9,

$$h_\mu(\Phi) = h_\mu(\Phi, \alpha) = -\int \log \mu_{\Phi^{-1}\alpha(w)}(\alpha(w)) \; d\mu(w),$$

and

$$\mu_{\Phi^{-1}\alpha(w)}(\alpha(w)) = \mu_{\alpha(\Phi w)}(\Phi(\alpha(w))) = \frac{\mu^u(\Phi(\alpha(w)))}{\mu^u(\alpha(\Phi w))} = e^{-\delta_\Gamma T} \frac{\mu^u(\alpha(w))}{\mu^u(\alpha(\Phi w))}.$$

Hence

$$h_\mu(\Phi) = \delta_\Gamma T - \int \log \frac{\mu^u(\alpha(w))}{\mu^u(\alpha(\Phi w))} \; d\mu(w) = \delta_\Gamma T,$$

from lemma 5.1.10. Since $\Phi = \varphi^T$, we get $h_\mu(\varphi) = \delta_\Gamma$. □

We can now proceed with the

Proof of theorem 5.1.1. Let us abbreviate μ_{BM} by μ, and assume the geodesic flow has no zero Lyapunov exponent on the nonwandering set. Since μ is supported on the nonwandering set, μ has no zero Lyapunov exponent and the last proposition gives $h_\mu = \delta_\Gamma$.

Now we prove that, for any invariant probability measure m, $h_m(\varphi^t) \leqslant \delta_\Gamma$. We can assume that m is ergodic, and so it is supported on the nonwandering set. Let α be a W^u-partition as in theorem 5.1.4, but this time, with respect to the measure m. α is not necessarily μ-measurable, but m-almost every atom $\alpha(w)$ is an open neighbourhood of w in $W^u(w)$, hence is Borelian, μ-measurable and has nonzero μ^u-measure. So we can set, for any μ-measurable set A,

$$\mu_{\alpha(w)}(A) := \frac{\mu^u(A \cap \alpha(w))}{\mu^u(\alpha(w))}.$$

In this way, α becomes "μ-measurable" and the same computation as before in proposition 5.1.11 gives

$$-\int \log \mu_{\Phi^{-1}\alpha(w)}(\alpha(v)) \; dm(w) = T\delta_\Gamma.$$

By Jensen inequality we get

$$T\delta_\Gamma - h_m(\Phi) = -\int \log \frac{\mu_{\Phi^{-1}\alpha(w)}(\alpha(w))}{m_{\Phi^{-1}\alpha(w)}(\alpha(w))}\, dm(w) \geqslant -\log\left(\int \frac{\mu_{\Phi^{-1}\alpha(w)}(\alpha(w))}{m_{\Phi^{-1}\alpha(w)}(\alpha(w))}\, dm(w)\right).$$

Finally, remark that

$$\begin{aligned}\int \frac{\mu_{\Phi^{-1}\alpha(w)}(\alpha(w))}{m_{\Phi^{-1}\alpha(w)}(\alpha(w))}\, dm(w) &= \int \left(\int_{\Phi^{-1}\alpha(w)} \frac{\mu_{\Phi^{-1}\alpha(w)}(\alpha(v))}{m_{\Phi^{-1}\alpha(w)}(\alpha(v))}\, dm_{\Phi^{-1}\alpha(w)}(v)\right) dm(w) \\ &= \int \left(\sum_{A\in\Phi^{-1}\alpha(w)} \mu_{\Phi^{-1}\alpha(w)}(A)\right) dm(w) \\ &\leqslant 1,\end{aligned}$$

so that $\delta_\Gamma \geqslant h_m(\Phi)$.

That proves that μ is a measure of maximal entropy. To prove uniqueness, we have to show that equality in the last inequality gives $m = \mu$. But this is the case if and only if there is equality in Jensen's inequality, that is,

$$\frac{\mu_{\Phi^{-1}\alpha(w)}(\alpha(w))}{m_{\Phi^{-1}\alpha(w)}(\alpha(w))} = 1,\ m - a.e. \tag{5.1}$$

Since α is generating, this implies that for m-almost any w, $\mu_{\alpha(w)} = m_{\alpha(w)}$. Let f be a continuous function with bounded support on HM, and denote by A_μ the set of $w \in HM$ such that

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(\Phi^k(w)) = \int f\ d\mu.$$

The ergodic theorem tells us that $\mu(A_\mu) = 1$. Furthermore, if $w \in A_\mu$, then by uniform continuity of f, the entire central stable manifold $W^{cs}(w)$ is contained in A_μ. Both facts and the local product structure of μ imply that A_μ has full μ^u_w-measure for *all* w. Thus, for m-almost every w (those such that $\alpha(w)$ is an open neighbourhood of w in $W^u(w)$), we have $\mu_{\alpha(w)}(A_\mu) = 1$, so that

$$m(A_\mu) = \int m_{\alpha(w)}(A_\mu)\ dm(w) = \int \mu_{\alpha(w)}(A_\mu)\ dm(w) = 1.$$

The ergodic theorem applied to m gives finally a set of full m-measure A_m, such that for all $w \in A_m$,

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(\Phi^k(w)) = \int f\ dm.$$

$A_m \cap A_\mu$ has now full m-measure, which implies $\int f\ d\mu = \int f\ dm$. Since f is arbitrary, we conclude that $m = \mu$. □

5.2 Ruelle inequality

We give a proof in our context of the famous Ruelle inequality and explicit the conditions under which it is actually an equality, following Ledrappier and Young [49]. It gives and efficient way to estimate entropies and will be essential to get the rigidity results of the next section.

5.2.1 A proof of Ruelle inequality

Theorem 5.2.1 (Ruelle inequality). *Let (Ω, d_Ω) be a Gromov-hyperbolic Hilbert geometry and $M = \Omega/\Gamma$ a quotient manifold. Let m be an invariant probability measure on HM. Then*

$$h_m(\varphi) \leqslant \int \chi^+ \ dm,$$

where $\chi^+ = \sum \dim E_i\ \chi_i^+$ denotes the sum of positive Lyapunov exponents.

(True only if the quotient is compact or convex-cocompact: this is the usual Ruelle inequality.)

Proof. This proof is inspired by the one appearing in [2] in the compact case. We can assume that m is ergodic. Recall that, since (Ω, d_Ω) is Gromov-hyperbolic, Ω is strictly convex and $\partial\Omega$ is $C^{1+\epsilon}$ for some $\epsilon > 0$. In particular, m has no zero Lyapunov exponent.
Let χ_1 be the smallest positive Lyapunov exponent and fix $\epsilon < \frac{\chi_1}{923}$. Let α be the leaf partition of theorem 5.1.4 induced by B_r. We endow each W^u-manifold with the metric d^u, generated by the restriction of the Finsler metric $\overline{F}$ on the W^u-manifold. For $d > 0$, define $U_d = \{w \in HM,\ \mathrm{diam}^u \alpha(w) > d\}$, where diam^u denotes the diameter with respect to d^u. Since $\alpha(w)$ is an open neighbourhood of w in $W^u(w)$, we have $\lim_{d\to 0} m(U_d) = 1$. Choose d such that $m(U_d) \geqslant 1 - \frac{\epsilon}{2}$.

Now, recall from section 5.1.2 the construction of the set $\Lambda_m(c)$, $c > 0$. Let

$$\Lambda_m(c, r) = \{w,\ W^u(w, r) \subset \Lambda_m(c)\}.$$

Choose $c > 0$ large enough to have $m(\Lambda_m(c, r)) > 1 - \frac{\epsilon}{2}$. Call $A_k = A_k(d, c) = \varPhi^{-k}(\Lambda_m(c, r) \cap U_d)$, so that $m(A_k) > 1 - \epsilon$.

We have, for $k \geqslant 1$,

$$\begin{aligned} h_m(\varPhi^k) &= \int -\log m_{\alpha(w)} \varPhi^k \alpha(w)\ dm(w) \\ &= \int_{A_k} -\log m_{\alpha(w)} \varPhi^k \alpha(w)\ dm(w) + \int_{HM \smallsetminus A_k} -\log m_{\alpha(w)} \varPhi^k \alpha(w)\ dm(w). \end{aligned}$$

The second term is less than $h_m(\varPhi^k)\epsilon$ (There is no reason to claim that. This was remarked by Barbara Schapira, thanks !). For the first one, we have

$$\begin{aligned} \int_{A_k} -\log m_{\alpha(w)} \varPhi^k \alpha(w)\ dm(w) &= \int_{A_k} \left(\int_{\alpha(w)} -\log m_{\alpha(w)} \varPhi^k \alpha(v)\ dm_{\alpha(w)}(v) \right) dm(w) \\ &\leqslant \int_{A_k} \log \sharp\{A \in \varPhi^k \alpha,\ A \subset \alpha(w), A \cap A_k \neq \emptyset\}\ dm(w). \end{aligned}$$

The set

$$\{A \in \varPhi^k \alpha,\ A \subset \alpha(w), A \cap A_k \neq \emptyset\}$$

consists of subsets $A = \Phi^{-k}(\alpha(\Phi^k v))$ for some $v \in \alpha(w) \cap A_k$. For such a v, we have $\Phi^k v \in U_d$, so that $\mathrm{diam}^u(\alpha(\Phi^k v)) > d$; furthermore, since $\Phi^k v \in \Lambda_m(c)$, we have

$$vol^u(A) \geqslant vol(\Phi^{-k}(W^u(\Phi^k v, d))) \geqslant \frac{1}{c} e^{-k(\chi^+ + \epsilon)} vol^u(W^u(\Phi^k v, d)),$$

where vol^u denotes the Busemann volume associated to the metric d^u (see section 1.4.4). Now recall that $\alpha(w) \subset W^u(w, r)$, so that $vol^u(\alpha(w)) \leqslant vol^u(W^u(w, r))$. Hence

$$\begin{aligned} \sharp\{A \in \Phi^k\alpha,\ A \subset \alpha(w), A \cap A_k \neq \emptyset\} &\leqslant \frac{vol^u(W^u(w,r))}{\min_{v \in \alpha(w)}\{vol^u(W^u(\Phi^k v, d))\}}\ ce^{k(\chi^+ + \epsilon)} \\ &\leqslant cDe^{k(\chi^+ + \epsilon)}, \end{aligned} \tag{5.2}$$

for some constant $D := D(r, d)$, as claimed by lemma 5.2.4 below. Finally,

$$h_m(\Phi) = \frac{1}{k} h_m(\Phi^k) \leqslant \frac{1}{k} \log(cD) + (\chi^+ + \epsilon) + \epsilon h_m(\Phi).$$

Let k go to $+\infty$ to get

$$h_m(\Phi) \leqslant \chi^+ + \epsilon(1 + h_m(\Phi)).$$

Since ϵ is arbitrarily small, we have the result. □

To prove the claim about volumes in inequality (5.2), we have to recall two results about the set X_n of convex proper open subsets of $\mathbb{RP}^n$. For $\delta \geqslant 0$, we let

$$X_n^\delta = \{\Omega \in X_n,\ (\Omega, d_\Omega) \text{ is } \delta\text{-hyperbolic}\}$$

and

$$X_{n,0} = \{(\Omega, x),\ \Omega \in X_n,\ x \in \Omega\},\ X_{n,0}^\delta = \{(\Omega, x),\ \Omega \in X_n^\delta,\ x \in \Omega\}.$$

The projective group $PGL(n+1, \mathbb{R})$ acts on each of these sets.

Theorem 5.2.2 (Benzécri, [9])**.** *The action of $PGL(n+1, \mathbb{R})$ on $X_{n,0}$ is proper and cocompact, that is, $X_{n,0}/PGL(n+1, \mathbb{R})$ is compact.*

Proposition 5.2.3 (Benoist, [6])**.** *Let $\delta \geqslant 0$. The set X_n^δ is a $PGL(n+1, \mathbb{R})$-invariant closed subset of X_n.*

Both results imply that the quotient $X_{n,0}^\delta / PGL(n+1, \mathbb{R})$ is compact, hence the expected

Lemma 5.2.4. *Let $\delta \geqslant 0$ and $r > 0$. There exist constants $v = v(r, \delta) > 0$ and $V = V(r, \delta) > 0$ such that, for any δ-hyperbolic Hilbert geometry (Ω, d_Ω) and $w \in H\Omega$,*

$$v < vol^u(W^u(w, r)) \leqslant V.$$

Proof. Consider the function

$$\begin{array}{rccl} f: & X_{n,0}^\delta & \longrightarrow & (0, +\infty) \\ & (\Omega, x) & \longmapsto & \max\{vol^u(W^u(w, r)),\ w \in H_x\Omega\}. \end{array}$$

This function is continuous and $PGL(n+1, \mathbb{R})$-invariant. Since $X_{n,0}^\delta / PGL(n+1, \mathbb{R})$ is compact, f is bounded: there exists $V > 0$ such that, for any $\Omega \in X_{n,0}^\delta$ and $w \in H\Omega$, $vol^u(W^u(w, r)) \leqslant V$.
The same can be done with the function $g : (\Omega, x) \longmapsto \min\{vol^u(W^u(w, r)),\ w \in H_x\Omega\}$ to get the lower bound. □

I guess Ruelle inequality should be true for all Hilbert geometries but the proof would be a bit more involved. Anyway, we do not really need it for the applications.

5.2.2 Sinai measures and the equality case

An invariant measure that achieves the equality in Ruelle inequality is called a **Sinai measure**. This is named under the name of Sinai because Sinai proved that equality occurs when the measure is a smooth measure. More generally, the theorem is the following:

Theorem 5.2.5 (Ledrappier-Young [49]). *Let (Ω, d_Ω) be a Gromov-hyperbolic Hilbert geometry and $M = \Omega/\Gamma$ a quotient manifold. Let m be a φ^t-invariant probability measure on HM. Then m is a Sinai measure if and only if it has absolutely continuous conditional measures on W^u-manifolds.*

(The proof works only if the nonwandering set is compact, that is, for compact and convex cocompact quotients; this is Ledrappier-Young result.)

Proof of theorem 5.2.5. We just give an idea of the proof, details can be found in [2] or [49]. It can be reduced to the case when m is ergodic. So let vol be the volume defined on HM by $\overline{F}$ and m be an ergodic invariant measure of the flow. Take a W^u-partition α as in theorem 5.1.4, such that

$$h_m = \int -\log m_{\Phi^{-1}\alpha(w)}(\alpha(w))\ dm(w). \tag{5.3}$$

If m has absolutely continuous unstable measures, then we can write $dm_{\alpha(w)} = f\ dvol_{\alpha(w)}$. Now, we can see that f must be proportional for $v \in \alpha(w)$ to the infinite product

$$f(v) = \prod_{n=1}^{+\infty} \frac{J^u(\Phi^{-n}v)}{J^u(\Phi^{-n}w)}, \tag{5.4}$$

with $J^u(v) = \det d_v\Phi^{-1}|_{E^u}$, which is well defined thanks to the $C^{1+\epsilon}$ regularity of the boundary, which implies $C^{1+\epsilon}$ regularity of the flow. (In fact, this regularity condition is not sufficient to ensure the existence of J^u as I discovered later; this is because of the noncompactness of the quotient; it works if the nonwandering set is compact but I don't know how to fix it in the general case.) Equation (5.4) now gives the equality.

For the converse, the argument is similar to the one used to prove theorem 5.1.1. Assume m is a Sinai measure, that is $h_m = \int \chi^+\ dm$. Let f be as in (5.4) and define a new Borel measure ν by setting $d\nu_{\alpha(w)} = f\ dvol_{\alpha(w)}$. In this way, $\nu = vol$ on the subalgebra $\mathcal{B}_\alpha$ of $\mathcal{B}$ which contains all unions of elements of α; for a Borelian B, the measure $\nu(B)$ is well defined by

$$\nu(B) = \int \nu_{\alpha(w)}(B)\ d\nu(\alpha(w)).$$

Then we can prove that $m = \nu$. We first check that

$$h_m = \int -\log \nu_{\Phi^{-1}\alpha(w)}(\alpha(w))\ dm(w).$$

By Jensen inequality and the fact that α is generating, we get that for m-almost every w,

$$\nu_{\alpha(w)} = m_{\alpha(w)},$$

which gives $dm_{\alpha(w)} = f\ dvol_{\alpha(w)}$. ∎

5.3 Entropy rigidities

5.3.1 Compact quotients

A pragmatic goal of this thesis was to distinguish Riemannian hyperbolic structures from non-Riemannian strictly convex projective ones by their entropies. For compact manifolds, a complete answer is given by theorem 5.3.3, which is the main result of the article [25].
The first step in the proof of this theorem is the following general rigidity result:

Proposition 5.3.1. *Let (Ω, d_Ω) be a Gromov-hyperbolic Hilbert geometry and $M = \Omega/\Gamma$ a quotient manifold. Assume there is a finite Bowen-Margulis measure on HM and denote by μ_{BM} the probability one. Then*

$$\delta_\Gamma \leqslant n-1,$$

with equality if and only if μ_{BM} is absolutely continuous.

(It is not true in this generality because the proof of Ruelle inequality does not work.)

Proof. Proposition 5.1.11 gives $h_{\mu_{BM}} = \delta_\Gamma$. Ruelle inequality implies that

$$h_{\mu_{BM}} \leqslant \int \chi^+ \, d\mu_{BM} = n-1 + \int \eta \, d\mu_{BM},$$

where η corresponds to the parallel transport, as in proposition 3.2.1.
As we saw from the Patterson-Sullivan construction (section 4.2.2), the Bowen-Margulis measure is flip-invariant, that is $\sigma * \mu_{BM} = \mu_{BM}$. We could also use the unicity of the measure of maximal entropy to prove it. Recall now from lemma 3.2.2 that η is antisymmetric to get

$$\int \eta \, d\mu_{BM} = 0,$$

and

$$\delta_\Gamma \leqslant n-1.$$

From theorem 5.2.5, equality occurs if and only if μ_{BM} has absolutely continuous unstable conditional measures. But this is equivalent to the absolute continuity of the Patterson-Sullivan measures, that is, to the absolute continuity of the whole measure μ_{BM}. □

The next lemma gives a criterion to apply proposition 5.3.1.

Lemma 5.3.2. *Let (Ω, d_Ω) be a Gromov-hyperbolic Hilbert geometry and $M = \Omega/\Gamma$ a quotient manifold. Assume there exists a probability Bowen-Margulis measure μ_{BM}. If Γ is Zariski-dense in $SL(n+1, \mathbb{R})$, then μ_{BM} is not absolutely continuous.*

(It works only for compact and convex cocompact quotients, for the same reason as in the proof of theorem 5.2.5.)

Proof. Assume $\mu := \mu_{BM}$ is absolutely continuous with respect to the volume vol defined by the metric $\overline{F}$. Call vol^s and vol^u the volumes defined by $\overline{F}$ on the stable and unstable manifolds. Take a W^u-partition α as in theorem 5.1.4. As in theorem 5.2.5, we can see that, on μ-almost every $\alpha(w)$, $\mu^u = f^u vol^u$, where $f^u(v)$, for $v \in \alpha(w)$, is proportional to the infinite product

$$\prod_{n=1}^{+\infty} \frac{J^u(\Phi^{-n}v)}{J^u(\Phi^{-n}w)}. \tag{5.5}$$

What is important is that f^u is continuous and $f^u > 0$. In the same way, we see that $\mu^s = f^s vol^s$ with f^s positive and continuous. This implies that μ itself satisfies $\mu = f vol$, with f positive and continuous on the support of μ. But the support of μ is the whole nonwandering set, so f is positive and continuous on the nonwandering set.

Now, consider the periodic orbit γ of length $l(\gamma)$ associated to the hyperbolic element $\gamma \in \Gamma$. Pick $w \in \gamma$. Since f is positive on the orbit γ, it implies that $d_w\varphi^{l(\gamma)}$ is a linear automorphism of T_wHM such that

$$|\det d_w\varphi^{l(\gamma)}| = 1.$$

Together with proposition 3.6.1, that implies

$$0 = \lim_{t\to+\infty} \frac{1}{t} \log|\det d_w\varphi^t| = 2\eta(\gamma) = 2(n+1)\frac{\log\lambda_0(\gamma) + \log\lambda_{p+1}(\gamma)}{\log\lambda_0(\gamma) - \log\lambda_{p+1}(\gamma)},$$

where $\lambda_0(\gamma)$ and $\lambda_{p+1}(\gamma)$ denote the biggest and smallest eingenvalues of γ. Thus, for any $\gamma \in \Gamma$, we have

$$\log\lambda_0(\gamma) + \log\lambda_{p+1}(\gamma) = 0,$$

or

$$\lambda_0(\gamma)\lambda_{p+1}(\gamma) = 1.$$

But, from theorem 1.2.a.β of [4], such an equation cannot occur for all hyperbolic elements $\gamma \in \Gamma$ if Γ is Zariski-dense. □

We can now state the

Theorem 5.3.3. *Assume $M = \Omega/\Gamma$ is compact. Then $\delta_\Gamma = h_{top} \leqslant n-1$, with equality if and only any of the following equivalent propositions is satisfied:*

- *M is Riemannian hyperbolic;*
- *the parallel transport on M is an isometry;*
- *the Bowen-Margulis measure is absolutely continuous.*

The last result which is useful to get the theorem is the following. It was shown by Benoist for cocompact groups, but his proof readily extends to get the

Theorem 5.3.4 (Y. Benoist, [5])**.** *Let $\Gamma \subset Isom(\Omega, d_\Omega)$ such that $\Lambda_\Gamma = \partial\Omega$. Then the Zariski-closure of Γ is either conjugated to $SO(n,1)$ or it is all of $SL(n+1,\mathbb{R})$.*

Proof of theorem 5.3.3. In this case of a compact manifold, μ_{BM} is exactly the measure of maximal of maximal entropy constructed by Bowen and Margulis, so $h_{top} = h_{\mu_{BM}}$. The equality $\delta_\Gamma = h_{top}$ is Manning's theorem 1.6.2. Of course, this is also a special case of theorem 5.1.1.

Now, recall that (Ω, d_Ω) is necessarily Gromov-hyperbolic, from Benoist's theorem 1.4.2. Proposition 5.3.1 gives

$$\delta_\Gamma \leqslant n-1,$$

with equality if and only if μ_{BM} is absolutely continuous. If the case M is Riemannian hyperbolic, μ_{BM} is actually the Liouville measure and there is equality. Otherwise, theorem 5.3.4 together with lemma 5.3.2 say that μ_{BM} cannot be absolutely continuous. The proposition about parallel transport is just what was proved in the course of the proof of lemma 5.3.2.

□

Together with proposition 1.6.1, we get the following

Corollary 5.3.5. *Let Ω be a divisible strictly convex set. Then*

$$h_{vol}(\Omega) \leqslant n-1,$$

with equality if and only if Ω is an ellipsoid.

The existence of divisible sets in all dimensions gives a lot of Hilbert geometries whose volume entropy is strictly between 0 and $n-1$. This statement is then a more precise answer to the conjecture 1.5.2 for divisible strictly convex sets.

5.3.2 Finite volume surfaces

I hoped to extend the last rigidity results to noncompact quotients and the last two chapters were the first steps to such extensions. General results are not available yet but, in the particular case of surfaces where we can understand the possible quotients, we get the following extension of theorem 5.3.3, whose proof is exactly the same (recall that (Ω, d_Ω) is Gromov-hyperbolic from Marquis' proposition 4.4.2):

Theorem 5.3.6. *Let $M = \Omega/\Gamma$ be a surface of finite volume. Then $\delta_\Gamma \leqslant 1$, with equality if and only if any of the following equivalent propositions is satisfied:*

- *M is Riemannian hyperbolic;*
- *the parallel transport on M is an isometry;*
- *the Bowen-Margulis measure is absolutely continuous.*

(This depends on previous results whose proofs are wrong...)
Together with theorem 4.4.1, this implies the following

Corollary 5.3.7. *Assume $\Omega \subset \mathbb{RP}^2$ admits a quotient of finite volume. Then*

$$h_{vol}(\Omega) \leqslant n-1,$$

with equality if and only if Ω is an ellipsoid.

Let me end this part with some remarks.

First of all, I thought it was possible to go further and to prove that, for any geometrically finite surface $M = \Omega/\Gamma$, we had $\delta_\Gamma \leqslant 1$, and that equality occurred if and only if M was Riemannian hyperbolic with finite volume. Indeed, it is known that if $M = \mathbb{H}^n/\Gamma$ is a geometrically finite manifold with infinite volume, then $\delta_\Gamma < n-1$, and so we could expect the same in our case. I still guess it is true, but it is not so easy, as we now see.
In $SL(3,\mathbb{R})$, the only infinite Zariski-closed subgroups are, up to conjugation, $SO(3)$, $SO(2,1)$ and $SL(3,\mathbb{R})$. Since $SO(3)$ is compact, the Zariski-closure $\overline{\Gamma}$ of an infinite discrete subgroup Γ of $Isom(\Omega, d_\Omega)$ can be either a conjugate of $SO(3,1)$ or $SL(3,\mathbb{R})$.

- If $\overline{\Gamma} = SL(3,\mathbb{R})$, lemma 5.3.4 applies and as before we get $\delta_\Gamma < 1$;
- If $\overline{\Gamma}$ is conjugated to $SO(2,1)$, then, that means Γ acts on some ellipsoid. In particular, the limit set lies on an ellipsoid. Nevertheless, that does not imply that the geometry is Riemannian hyperbolic, because the limit set is in general not the whole of $\partial\Omega$. So Ω has a lot of points in common with an ellipsoid but that is all we know.

Let us recall that, when $M = \Omega/\Gamma$ is compact, the critical exponent is exactly the exponential growth rate of numbers of closed geodesics of length at most t:

$$\delta_\Gamma = \lim_{t\to+\infty} \frac{1}{t} \log \sharp\{\gamma \in \Gamma,\ \gamma \text{ hyperbolic and } l(\gamma) \leqslant t\}.$$

If the same were true for geometrically finite surfaces, then δ_Γ would depend only on the group Γ and not on Ω, hence we could conclude from the fact that Γ acts on some $\mathbb{H}^2$. But I do not know if this remains true...

Second, we could also want to distinguish Riemannian hyperbolic structures and non-Riemannian ones on surfaces of infinite volume by some dynamical invariant. But in this case, topological entropy is clearly not what we have to look at. Take for example a convex cocompact hyperbolic surface. It is known that its topological entropy depends on the hyperbolic structure and can take all the values which are strictly between 0 and 1. Thus, we cannot expect a result like theorem 5.3.6: there would be some non-Riemannian structures whose topological entropy would be bigger than the topological entropy of some hyperbolic structure. So a new rigidity result has to be formulated in this context.

5.4 Continuity of entropy

We finish this section, chapter and thesis by the following proposition, which asserts that the entropy of a compact manifold or a finite volume surface varies continuously with the structure. By varying the structure, we mean the following. Take an abstract smooth compact manifold M, which admits a strictly convex projective structure $M_0 = \Omega_0/\Gamma_0$. This means we are given a developing map $dev_0 : \tilde{M} \longrightarrow \Omega_0$, which is a diffeomorphism from the universal cover of M to Ω_0, and a representation $\Gamma_0 = \rho_0(\pi_1(M))$ of the fundamental group of M as a faithful and discrete subgroup of $PGL(n+1,\mathbb{R})$. Remark that the convex set Ω_0 itself is indeed determined by this representation, since the limit set Λ_Γ is the whole of $\partial\Omega$. Endow the set $Hom(\pi_1(M), PGL(n+1,\mathbb{R}))$

of representations with the compact-open topology, and the set of maps $\tilde{M} \longrightarrow \mathbb{RP}^n$ with the topology of uniform convergence. A continuous deformation of the structure is a path $(dev_\lambda, \rho_\lambda)$ of convex projective structures which is continuous with respect to these topologies. The same can be done for deformations of finite volume convex projective structures on a surface M.

Proposition 5.4.1. *Assume $M_0 = \Omega_0/\Gamma_0$ is compact (resp. a surface of finite volume). Let $M_\lambda = \Omega_\lambda/\Gamma_\lambda$, $\lambda \in [-1,1]$ be a continuous deformation of M into compact manifolds (resp. finite volume surfaces). Then the function $\lambda \longmapsto \delta_{\Gamma_\lambda}$ is continuous.*

Proof. Let us do the proof in the compact case. Let $(\rho_\lambda, dev_\lambda)$, $\lambda \in [-1,1]$ be the considered deformation of (ρ_0, dev_0). These structures provide Finsler metrics F_λ on the abstract manifold M. These metrics vary continuously with λ in the following sense:

$$\lim_{\lambda\to 0} \sup_{TM\smallsetminus\{0\}} \frac{F_\lambda}{F_0} = 1.$$

For let T^1M the unit tangent bundle for F_0. Since T^1M is compact[1] and $\lambda \mapsto dev_\lambda$ is continuous,

$$\lim_{\lambda\to 0} \sup_{T^1M} |F_\lambda - F_0| = 0.$$

Moreover $\min_{T^1M} F_0 > 0$, hence

$$\lim_{\lambda\to 0} \sup_{T^1M} |\frac{F_\lambda}{F_0} - 1| = 0.$$

Homogeneity gives the result, that is there exist reals $C_\lambda \geqslant 1$ such that $\lim_{\lambda\to 0} C_\lambda = 1$ and

$$C_\lambda^{-1} \leqslant \sup_{TM\smallsetminus\{0\}} \frac{F_\lambda}{F_0} \leqslant C_\lambda.$$

Denote by $\tilde{d}_\lambda$ the associated distances on $\tilde{M}$. Let $x, y \in \tilde{M}$, and c_λ be the geodesic from x to y for the metric $\tilde{d}_\lambda$, such that $\int \tilde{F}_\lambda(c'_\lambda(t))\ dt = \tilde{d}_\lambda(x,y)$. Then

$$C_\lambda^{-1} \leqslant \frac{\int \tilde{F}_\lambda(c'_\lambda(t))\ dt}{\int \tilde{F}_0(c'_\lambda(t))\ dt} \leqslant \frac{\tilde{d}_\lambda(x,y)}{\tilde{d}_0(x,y)} \leqslant \frac{\int \tilde{F}_\lambda(c'_0(t))\ dt}{\int \tilde{F}_0(c'_0(t))\ dt} \leqslant C_\lambda.$$

Thus for any $x, y \in \tilde{M}$,

$$C_\lambda^{-1} \leqslant \frac{\tilde{d}_\lambda(x,y)}{\tilde{d}_0(x,y)} \leqslant C_\lambda.$$

From that we clearly get $\tilde{B}_\lambda(x,R) \subset \tilde{B}_0(x, C_\lambda R)$. Hence

$$\delta_{\Gamma_\lambda} = \limsup_{R\to\infty} \frac{1}{R} \sharp\{g \in \pi_1(M),\ gx \in \tilde{B}_\lambda(x,R)\} \leqslant \limsup_{R\to\infty} \frac{1}{R} \sharp\{g \in \pi_1(M),\ gx \in \tilde{B}_0(x, C_\lambda R)\} = C_\lambda \delta_{\Gamma_0}.$$

Similarly, $C_\lambda^{-1}\delta_{\Gamma_0} \leqslant \delta_{\Gamma_\lambda}$. That gives the continuity. □

[1] In the case of a finite volume surface, one has to use the fact that the geometry is controlled in the cusps.

Postface et remerciements

> Si ça ne tenait qu'à moi, il n'y aurait rien de définitif. Cette thèse se termine là, mais elle est inachevée; il y a certainement des erreurs de mathématiques et l'anglais, soyons indulgent, n'y est pas très bon. J'aurais pu faire beaucoup plus de figures, mais c'est très long de faire des figures alors il n'y a que celles qui sont indispensables ou que j'ai réussi à faire.
> Dans cette partie, on peut parler de tout. Du temps, des choses et des gens. C'est la partie qui est la plus difficile à commencer. On ne sait pas trop où elle nous mène, mais on sait qu'il y a des passages obligés. Remercier mes parents par exemple[2].
> Je vais essayer de m'en tenir à ma thèse, aux mathématiques, à l'université. C'est déjà bien trop en fait, cela fait beaucoup de choses, de temps, de gens.

En faisant des mathématiques avec d'autres mathématicienNEs, je suis entré dans la communauté mathématique. Elle ne se limite pas du tout aux mathématicienNEs, mais les mathématiques en sont bien le nœud. Son but est de faire progresser les mathématiques, de découvrir de nouvelles choses en mathématiques. En fait, je préfère penser que son but est juste de faire des mathématiques, mais c'est ma façon de voir. Qu'il n'y ait pas de notion de progrès, ça me rassure...
D'un point de vue institutionnel, c'est mon appartenance à une université qui compte, et je suis ainsi membre de la communauté universitaire. Je l'étais avant en tant qu'étudiant, mais j'y ai changé de statut (tout en gardant les avantages de celui d'étudiantE auprès de la société civile). En changeant de statut, j'ai eu droit à tout plein de choses, sûrement parce que je suis devenu plus important pour/dans la communauté. J'ai eu droit à un bureau, à une clé du bâtiment de mathématiques, de la bibliothèque et du garage à vélos, à imprimer et photocopier par milliers, à rendre mes livres avec 3 mois de retard; je n'ai pas usé de ce dernier droit mais par contre, j'ai usé de celui d'étaler sur un livre de la bibliothèque du coulis de fruits de houx: Myriam Pepino m'a même dit que c'était la première fois que quelqu'unE faisait ça et du coup, grâce à moi, elle connaît l'effet du fruit de houx sur les pages d'un livre. Avec cette anecdote, ce sont toutes les personnes qui travaillent à l'IRMA et à l'UFR de maths que je voudrais remercier, car sans elles on ne pourrait pas faire de maths; en particulier, je remercie Yvonne Borell grâce à qui j'ai eu à traverser le couloir et non le campus pour imprimer ma thèse.
Encore une fois, j'aime croire que l'université est là juste pour penser, pour faire des sciences, au sens large: dures, molles, salées, demi-sel, avec des cristaux de sel. Bien sûr, je sais que c'est faux, que la science et le savoir sont une industrie depuis belle lurette, et qu'ils servent un idéal de vie, bourgeois et capitaliste. C'est aussi pour ça que je n'aime pas le progrès.

[2] C'est fait...

Que j'entre dans ces communautés, ça veut dire qu'on me reconnaît à un moment y prendre une part *active*. C'est moins le cas quand on est unE vraiE étudiantE, à mon sens ça ne devrait pas mais c'est ainsi.
Pour évoluer dans ces communautés, il y a des règles, des règles claires, écrites, dites de droit, et puis d'autres, implicites. J'ai passé beaucoup de temps à apprendre des règles depuis... le début. C'est que j'ai dû faire partie de plein de communautés. L'État, la famille, l'école, le sport, l'extrême-gauche, la consommation... A chaque fois, il y a des individus pour vous montrer les règles du jeu de la communauté, et pour en décoder les coutumes.
C'est d'abord pour avoir joué ce rôle pendant ma thèse que je remercie Patrick. Je le remercie d'avoir pu discuter d'un peu tout et n'importe quoi, de m'avoir dit là où il fallait faire attention, de m'avoir conseillé sans m'obliger à suivre ses conseils, de m'avoir dit sa façon de voir et faire les choses et d'avoir accepté que j'ai la mienne. C'est appréciable de pouvoir discuter sans devoir être d'accord. Par exemple, c'est lui qui m'a conseillé de parler de tout, du temps, des choses et des gens seulement à la fin de cette thèse; sinon, je n'aurais pas résisté à en cacher des morceaux dans l'introduction.
Je le remercie aussi d'avoir pu étirer ses journées jusqu'à 26 ou 27 heures pour pouvoir faire des maths. C'est d'abord lui qui m'a fait découvrir ce dont on parle dans cette thèse, et c'est lui qui m'a lancé sur les bonnes pistes au début. En ça je crois que j'ai eu de la chance.

J'étais partagé pendant cette thèse. Entre Strasbourg et Bochum. À Bochum, j'ai appris beaucoup de choses en discutant avec Gerhard. Le sujet de cette thèse est plus éloigné de son travail, mais sa connaissance du monde riemannien courbé négativement m'a permis de prendre de nouvelles directions, de comprendre de nouveaux outils. Au final, il y a plein de choses dans cette thèse qui ne sont que des adaptations du riemannien... Je profite qu'on est à Bochum pour remercier Ursula Dzwigoll, qui m'a bien aidé lorsque j'étais là-bas.

Pendant un doctorat de mathématiques, on travaille souvent toutE seulE. C'est normal, je crois, qu'il y ait du temps pour penser toutE seulE dans la recherche. Bien entendu, on n'est en fait jamais seulE, encore moins aujourd'hui qu'on a internet et des bibliothèques géantes. Et au fond, il n'est pas si facile de dire qui a joué quel rôle dans l'élaboration d'une idée.
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Ludovic, c'est la personne avec qui j'ai fait le plus de maths pour de vrai, ce qui est possible parce qu'on travaille sur la même chose... J'ai bien aimé travailler avec lui, j'aime toujours bien et c'est tant mieux parce qu'on n'a pas fini. On ne sait toujours pas si l'on découvre ou si l'on invente les mathématiques, mais peu importe.

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Caminante, son tus huellas	Marcheur, ce sont tes traces
el camino, y nada mas;	le chemin, et rien de plus;
caminante, no hay camino,	marcheur, il n'y a pas de chemin,
se hace camino al andar.	le chemin se fait en marchant.
Al andar se hace camino,	En marchant se fait le chemin,
y al volver la vista atras	et lorsqu'on regarde derrière soi
se ve la senda que nunca	on voit le sentier que jamais plus
se ha de volver a pisar.	on ne foulera.
Caminante, no hay camino,	Marcheur, il n'y a pas de chemin,
sino estelas en la mar.	mais seulement des sillons laissés dans la mer.

J'ai eu la chance pendant mon doctorat de participer à divers événements mathématiques, au cours desquels j'ai pu rencontrer et discuter avec de nombreuSES mathématicienNEs.
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"Le peuple n'a que ce qu'il prend"

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